Philosophy of Mathematics

Wittgenstein himself emphasized in 1944 (in a biographical entry) that his “chief contribution has been in the philosophy of mathematics” (Monk 1990, 466)

We are led astray by our expectations that language always functions in the same way, for example that all words are names that refer to objects

We must take note of how words like ‘number’, ‘infinite, ‘series’, ‘proof’, etc. function when used in mathematics, i.e. describe their “grammar”

“Knowledge in mathematics: Here one has to keep on reminding oneself of the unimportance of the ‘inner process’ or ‘state’ and ask ‘Why should it be important?’ What does it matter to me? What is interesting is how we use mathematical propositions.”

On Certainty §38, p. 7e

Mathematics in the *Tractatus*

- The context of Wittgenstein's discussion is given by the “Grundlagenkrise” in mathematics that was prompted by the development of mathematics in the 19th century

- We saw earlier three philosophical responses to this crisit: 1) Empiricism and Psychologism (Mill), 2) Formalism (Hilbert), and 3) Logicism (Russell and Frege)

- In Wittgenstein’s early work, the mathematical logic of Frege and Russell is a central inspiration, but the treatment of mathematics is not particularly central to that work. Wittgenstein is more interested in logic, thought, and language.

TLP : Mathematics is “a method of logic” (6.234) or “a logical method” (6.2)

This sounds like logicism, but it isn’t.

- “The propositions of mathematics are equations, and therefore pseudo-propositions.” (TLP 6.2) i.e. somewhat similar to tautologies.

Equations state identities between names. But in a *Begriffsschrift*, there cannot be two names for the same object. So all equations are uninformative. Since they are not even built up out of logical vocabulary, they aren’t even tautologies. But there is something “ok” about them. (cf. Kremer article)

- “The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics.” (6.22)

- “What is essential about an equation is that it is not necessary to show that both expressions, which are connected by the sign of equality, have the same meaning; for this can be perceived from the two expressions themselves.” (6.232)

- Wittgenstein was always critical of the (Platonist) view that numerals are signs that refer to numbers as their referents. He thinks construing numerals as the names of numbers is confused.

- TLP 6.031 “The theory of classes is completely superfluous in mathematics.”

- Both Frege and Russell identified numbers as objects, classes of classes. For example, 2 is the class of all 2-membered classes. The numeral ‘2’ refers to this object.

- TLP 6.021 “A number is the exponent of an operation.”

- What does this mean?

TLP 5.22 “An operation is the expression of a relation between the structures of its result and of its bases.”

TLP 5.23 “The operation is what has to be done to the one proposition in order to make the other out of it.”

So, if we start with P and apply the truth operation of negation to it, we get ~P. If we go on to apply the truth operation of negation to that base, we get ~ ~P.

We negate a proposition and its negation: We carry out the operation *twice*.

And so on…

Rules in TLP: 5.2523 “The concept of successive applications of an operation is equivalent to the concept ‘and so on’.”

TLP 6.0. “The general form of a truth function is [p, ζ, N(ζ)].
This is the general form of a proposition.”

TLP 6.001 “What this says is just that every proposition is a result of successive applications to elementary propositions of the operation N(ζ).”

What does it mean to be an ***exponent*** of an operation? It is repeating the operation ***n*** times.

“Daddy, give me 4 cookies!” (NB! “Taking a cookie” is NOT a logical operation!)

“I will not take a cookie from the cookie jar for you for you! You get 0 cookies.”

Or,

“I take a cookie from the jar for you, and I take a cookie from the jar for you, and I take a cookie from the jar for you, and I take a cookie from the jar for you.”

So, I take 4 cookies from the jar for you. But “4” doesn’t *refer* to the object 4. It merely indicates the repetition of a counting procedure !!!! times.

The mathematical “proposition” “3+2=5” can be read as the result of taking a cookie, and taking a cookie, and taking a cookie. And then later taking a cookie and taking a cookie and then seeing how many cookies I end up with.

TLP 6.21-6.211 “A proposition of mathematics does not express a thought.

Indeed in real life a mathematical proposition is never what we want. Rather, we make use of mathematical propositions only in inferences from propositions that do not belong to mathematics to others that likewise do not belong to mathematics. (In philosophy the question, ‘What do we actually use this word or this proposition for?’ repeatedly leads to valuable insights.”

The meaning of mathematical propositions, and the number terms in them, are connected with ***doing*** something. The propositions don’t correspond to facts and the numerals don’t name numbers.

I can repeat the operation of taking cookies out of the jar as many times as I want by just ***going on the same way***, and so on.

So we see that Wittgenstein’s whole account of numbers and mathematics is tied to the intelligibility of the truth functionality of language and this idea of “and so on.”

The Middle Period (1929-1935)

- W’s return to philosophy in 1929 is preceded by his attending a lecture by the Dutch mathematician L.E.J. Brouwer on the Foundations of Mathematics.

- Brouwer was the main advocate of “Intuitionism”

Brouwer claims that our “primodial intuition of time” (cf. Kant) means our fundamental concepts of mathematics are acquired once we can recognize change.

We construct the mathematical world out of our concepts; it’s not a matter of trying to bring our conceptual resources to bear on a mind-independent mathematical reality.

From this follows that mathematics cannot intelligibly comprehend the properties of infinite sets; only what can be shown to be provable can be justifiably asserted (there are no “undecidable propositions” in math); i.e., mathematics must be seen as resting on (finite) human activity.

- Wittgenstein is sometimes mistakenly thought to be an intuitionist. But there *is* some overlap between his thought and Brouwer’s, especially on finitude.

- In this period he also read works by Weyl, Skolem, Ramsey and possibly Hilbert

- Also discussions with Ramsey and members of the Vienna Circle centered on mathematics

- The main rival to intuitionism in the debate about the Foundations of Mathematics at this time was formalism, whose most important advocate was David Hilbert.

- According to the formalist mathematics is not, or need not be, *about* anything, or anything beyond typographical characters and rules for manipulating them.

- Mathematics is a science of formal systems. The essence of mathematics is the manipulation of characters or “symbols”. A mathematical statement has no meaning but its symbols, regarded as physical objects, exhibit a structure that has applications. These are only restricted by the demand of consistency of the axioms of mathematics.

Wittgenstein on the Need for Foundations

“What does mathematics need a foundation for? It no more needs one, I believe, than propositions about physical objects—or about sense impressions, need an *analysis*. What mathematical propositions do stand in need of is a **clarification of their grammar**, just as do those other propositions.

The *mathematical* problems of what is called foundations are no more the foundation of mathematics for us than the painted rock is the support of a painted tower.

‘But didn't the contradiction make Frege’s logic useless for giving a foundation to arithmetic?’ Yes, it did. But then, who said that it had to be useful for this purpose?”

- RFM VII §16

Language as a Calculus

- Calculus model: mathematics consists of self-contained calculi (mathematics also functions as a model for natural language)

- Mathematical propositions are to be understood as rules for these calculi.

- An equation is a “rule of grammar” that fixes the meaning of symbols that occurr in it.

- A mathematical calculus does not need an extra-mathematical application.

(*PR* §109; *WVC* 105)

- Similarly, language operates like a calculus with rules and uses sharply separated.

- Different systems of rules, and so different calculi, are possible. It is a matter of convenience and convention which one we adopt. Logic loses its all-encompassing “purity”.

Later Wittgenstein's Philosophy of Mathematics

- Language game view (in contrast to the earlier emphasis on ”propositions”)
- Mathematics can be seen as a "motley of techniques of proof"
- There is a *family resemblance* for instance among what we call "number" or "mathematics"
- "Meaning as use": the language games of mathematics must be seen in connection with the use of their signs outside mathematics, as part of an activity in "the stream of life" or as part of a "form of life"
- A mathematical 'proposition' functions as if it were an empirical proposition "hardened into a rule"

Dummett’s Wittgenstein

- Wittgenstein’s philosophy of mathematics represents an anti-realism and a “*full-blooded conventionalism”*

- This means that for him the logical necessity of any necessary truth is “the direct expression of a linguistic convention”

- “Necessity” is just an arbitrary decision or convention to treat certain propositions as necessary

“**Platonism**”

“I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our ’creations,’ are simply the notes of our observations.” G. H. Hardy, *A Mathematician's Apology* (London 1941)

- There is a realm of necessary facts independent of human thought (the “world of mathematical objects”)
- Mathematical propositions describe this realm
- These facts may ***outrun*** our ability to get access to them by proofs (the existence of “undecidable propositions” in mathematics)

Exampels of modern day Platonists in philosophy of mathematics:

- Frege
- Cantor
- Hardy
- Gödel

W thinks Platonism is misleading or even dangerous

Why?

- It distorts our understanding of what mathematics and mathematical activity is
- It gives mathematics a false inexorability and refuses to see it as a human activity and as dependent on what human beings actually do.

Goldbach's conjecture: "Every even integer greater than 2 can be expressed as the sum of two primes."

Is this a mathematical proposition?

Hardy: "Goldbach's conjecture is a proposition, and that is why I can believe it is true". (cf. BB, 14)

Wittgenstein:

When Hardy says he believes Goldbach's theorem, I would ask him what his belief in this theorem led him to. What does he do? It may have led him to attempts to prove it, which shows that *some* meaning attaches to the theorem inasmuch as these activities would not have been caused by another theorem.

- It is misleading to consider Goldbach's conjecture to be a mathematical proposition because we do not know how to decide it, i.e., we do not know how to make it either proved (true) or refuted (false)
- “What ‘mathematical questions’ share with genuine questions, is simply that they can be answered.”

- Wittgenstein wants to combat the Platonist picture that mathematics concerns “the natural history of mathematical objects” and that mathematics is a kind of natural science dealing with infinite extensions and other “mysteries of the mathematical world”.

- His “anthropological perspective” on mathematics is a part of this.

- But how are we then to understand such unproved mathematical “propositions” as Goldbach's conjecture?
- For instance as “signposts for mathematical research, stimuli for mathematical constructions”

Number is a family resemblance concept

—And I shall say: ‘games’ form a family.

> And for instance the kinds of number form a family in the same way. Why do we call something a “number”? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. PI §67

- So, natural numbers, real numbers, transfinite numbers, and imaginary numbers don’t *necessarily* have a common essence: they make up a family resemblance concept.

- Wittgenstein hated the idea that “number” had an essence: and some mathematicians hate him right back!

And So is Mathematics

- Mathematics can be viewed as a "family of activities for a family of purposes" (RFM V§15) or as "ein buntes Gemisch" ("motley"):

- I should like to say: mathematics is a MOTLEY of techniques of proof.—And upon this is based its manifold applicability and its importance. RFM III §46

- I want to give an account of the motley of mathematics. RFM III §48

- [N]ew types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* of this from the changes in mathematics.) PI §23

- The centrality of application in mathematics:

I want to say: It is essential to mathematics that its signs are also employed in *mufti* [*im Zivil*].

It is the use outside mathematics, and so the *meaning* [*Bedeutung*] of the signs, that makes the sign-game into mathematics. (*RFM* V, §2, 1942) (cf. *LFM* 140–141, 169–70)

In a certain sense it is not possible to appeal to the meaning of the signs in mathematics, just because it is only mathematics that gives them their meaning. (*RFM* V, §16)

Continuity in Later Wittgenstein's Philosophy of Mathematics

- "Mathematical propositions" are *not real propositions*
- "Mathematical truth" is essentially *non-referential*
- We *invent* mathematical calculi and we expand mathematics by calculation and proof
- Though we learn from a proof that a theorem *can* be derived from axioms by means of certain rules in a particular way, it is *not* the case that this proof-path must pre-exist our construction of it (there to be *discovered*)

Problems with Rules in TLP -

- The account of number is *finitistic*. It can’t account for transfinite numbers.
- He came to be suspicious of his own account of “and so on”, even with regard to finite numbers.

Problems with Rules in the Middle Period –

Are formal rules really as pervasive in our life with language to make it plausible that all of our various uses consist of self-contained calculi, or grammars, governed by rules at every step?

Challenges in the Remarks on Rule-Following – PI §§ 138-242

Keep the family resemblance critique of essentialism that precedes these remarks.

Show that ***even with arithemtic***, meaning is connected to use and practice.

The aim is **not** to show that grammar is “governed” by rules, or that uses/practices are “governed” by rules (they’re not: cf. PI §68), but that even rule-following in arithmetic is part of a practice.

This would put a nail in the **Platonist** coffin once and for all (If Platonism doesn’t work here, it won’t work anywhere.)