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Wittgenstein and Mathematics

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PI, preface

The thoughts which I publish in what follows are the precipitate of philosophical investigations which have occupied me for the last sixteen years. They concern many subjects: the concepts of meaning, of understanding, of a proposition, of logic, **the foundations of mathematics**, states of consciousness, and other things.

Mathematics and the PI

- The earliest version of the PI contained a large part concerned with mathematics as a direct continuation of the discussion about rule-following
  - the *Frühfassung* (the Early Draft) of the PI (1937–8) consisted of TS 220 (corresponding roughly to PI §§1–189(a)) and TS 221 (corresponding roughly to RFM I).
- This was later (1944) dropped in favour of the sections dealing with private language
- W continued working on the philosophy of mathematics parallel to his work on the PI
- Some of these manuscripts are (partly) published as RFM

Text sources

RFM ***Remarks on the Foundations of Mathematics / Bemerkungen über die Grundlagen der Mathematik*** (eds G. H. von Wright, R. Rhees, G. E. M. Anscombe)(3rd ed.1978)

LFM Diamond, Cora, (ed.) ***Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*** (1976)

Dummett's Wittgenstein

- Ws philosophy of mathematics represents an anti-realism and a "*full-blooded conventionalism"*
- This means that for him the logical necessity of any necessary truth is "the direct expression of a linguistic convention"
- "necessity" is just an arbitrary decision or convention to treat certain propositions as necessary

Wittgenstein and mathematics

- In Ws early work, mathematical logic is a central inspiration, but the treatment of mathematics is not particularly central
- However, W's "return to philosophy" in 1929 is preceded by his attending a lecture by the Dutch mathematician L.E.J. Brouwer
- In this period he also read works by Brouwer, Weyl, Skolem, Ramsey and possibly Hilbert
- Also discussions with Ramsey and members of the Vienna Circle centered on mathematics

Wittgenstein and mathematics

- In "the middle period" mathematics is of central importance to W's work and his thoughts about language
- W also lectures on "Philosophy for Mathematicians" (1932-1933) and on "The Foundations of Mathematics" (1939)
- PI was originally largely concerned with the philosophy of mathematics rather than the philosophy of psychology

- W himself emphasized in 1944 (in a biographical entry) that his "chief contribution has been in the philosophy of mathematics" (Monk 1990, 466)
- Wittgenstein's discussions on the philosophy of mathematics are an integral part of his later thought, and (at least up to 1944) a large part of his writings are directly about mathematics
- W did not consider mathematics to be a separate subject matter, but part of our "everyday ordinary language"

Psychology and mathematics

Is there any connection between the philosophy of psychology and the philosophy of mathematics? Obviously W seems to think there is.

An investigation entirely analogous to our investigation of psychology is possible also for mathematics. It is just as little a *mathematical* investigation as the other is a psychological one. It will *not* contain calculations, so it is not, for example, formal logic. It might deserve the name of an investigation of the 'foundations of mathematics'.

PI II, xiv (PPF §372)

The confusion and barrenness of psychology is not to be explained by calling it a "young science"; its state is not comparable with that of physics, for instance, in its beginnings. (Rather with that of certain branches of mathematics. Set theory.) For in psychology there are experimental methods *and conceptual confusion.* (As in the other case conceptual confusion and methods of proof.)

The existence of the experimental method makes us think we have the means of getting rid of the problems which trouble us; but problem and method pass one another by.

PI II, xiv (PPF §371)

Finitism and behaviorism are quite similar trends. Both say, but surely, all we have here is…. Both deny the existence of something, both with a view to escaping from a confusion.

RFM III §62

Finitism: only those entities may be admitted to mathematics that can be constructed in a finite number of steps, and only those propositions entertained whose truth can be proved in a finite number of steps

We might say that formalism in mathematics is behaviourism in mathematics. I could draw “2” and say “That is the number 2.” This is exactly the same as pinching and saying “This is pain.” – Mathematicians say “Surely it is not just the numeral, it is something more”. (LSD 111)

Morale of discussion of psychological concepts

- Neither the explanations nor the uses of such concepts have the formal simplicity and uniformity we naturally expect (on account of their "surface grammar")
- We are lead astray by our expectations that language always functions in the same way, for example that all words are names that refer to objects
- Especially when we consider our ways of talking about the ”inner” and the ”outer”, we must take note of how words such as ‘pain’, ‘thinking’, ‘anger’ function in our language

And analogously…

- We must take note of how words like 'number', 'infinite', 'series', 'proof', etc. function when used in mathematics, i.e. describe their "grammar"
- An investigation of the actual use of psychological/ mathematical concepts reveals complex and variegated patterns
- "Surveyable representations" are needed to clarify and describe those uses (and point out cases where we a mislead by them), and thus describe "the foundations of mathematics"

How are the investigations in philosophy of psychology and philosophy of mathematics related?

- They are joint by a criticism of two presuppositions that inform much of philosophy (often tacitly):
  - every meaningful word is a name
  - every sentence is a description
- The investigation of our mathematical and psychological concepts each clarify how this misconception about how language works shapes and distorts a major branch of philosophy.

Against mentalism (psychologism) AND Platonism in mathematics

Knowledge in mathematics: Here one has to keep on reminding oneself of the unimportance of the 'inner process' or 'state' and ask "Why should it be important?" What does it matter to me? What is interesting is how we use mathematical propositions.

On Certainty §38, p. 7e

The foundations of mathematics

- The context of Wittgenstein's discussion is given by the "Grundlagenkrise" in mathematics that was prompted by the development of mathematics in the 19th century
  - new forms of mathematics , eg. non-euclidic geometry, infinitesimal calculus, transfinite set theory, etc. challenged the received conception of math as a "science of quantity"
- Frege: *Grundgesetze der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl (1884)* was an attempt to give mathematics a foundation in formal logic by giving a purley logical definition of number (logicism)
- However, Russell's paradox (1903) showed that Frege's definition contained contradictions

The foundations of mathematics

- Russell's theory of types was an (ultimately unsuccessful) attempt to rescue the logicist program (Russell and Whitehead: *Principia Mathematica)*
- The "Grundlagenkrise" developed into the "Grundlagenstreit" between formalism (Hilbert) and intuitionism (Brouwer) in the 1920's
- In TLP W rejects the main claim of logicism, i.e. that logic is more fundamental than mathematics
- In his middle and late period W comments on, but does not take sides in, the fundamental issue between formalism and intuitionism (he is critical of both, though both contain elements he is sympathetic with)

Logicism

Logicism claims that all of mathematics can be deduced from logic; this accounts for the unassailability of mathematics

- The "Grundlagenkrise" begins with the problems of Frege's logicistic program: Russell's paradox (1903)

(simplified) logicistic definition of "1+1=2"

"$\forall x\ \forall y\ (((x \in 1) \wedge (y \in 1)) \equiv ((x \cup y) \in 2))$"

This is considered a statement of identity that is *logically true*

Logicism

- Russell's attempt to rescue logicism by the "theory of types" fails because now logic is no longer self-contained:
  - Russell's and Whitehead's *Principia Mathematica* thus builds upon axioms that are *not* logically self-evident, nor provable in the system

Wittgenstein thought this shows the hopelessness of the task:

TLP 6.1232 Logical general validity, we could call essential as opposed to accidental general validity, e.g. of the proposition "all men are mortal". Propositions like Russell's "axiom of reducibility" are not logical propositions, and this explains our feeling that, if true, they can only be true by a happy chance.

Intuitionism

Brouwer claims that our "primodial intuition of time" (cf. Kant) means our fundamental concepts of mathematics are acquired once we can recognize change.

We construct the mathematical world out of our concepts; it's not a matter of trying to bring our conceptual resources to bear on a mind-independent mathematical reality.

Intuitionism

From this follows that mathematics cannot intelligibly comprehend the properties of infinite sets; only what can be shown to be provable can be justifiably asserted (there are no "undecidable propositions" in math); i.e., mathematics must be seen as resting on (finite) human activity.

W on intutitionism

Intuitionism comes to saying that you can make a new rule at each point. It requires that we have an intuition at each step in calculation, at each application of the rule; for how can we tell how a rule which has been used for fourteen steps applies at the fifteenth? – And they go on to say that the series of cardinal numbers is known to us by a ground-intuition-that is, we know at each step what the operation of adding 1 will give. We might as well say that we need, not an intuition at each step, but a decision. – Actually there is neither. You don't make a decision: you simply do a certain thing. It is a question of a certain practice.

Intuitionism is all bosh-entirely. (LFM, 237)

Formalism

Mathematics is a science of formal systems. The essence of mathematics is the manipulation of characters or "symbols". A mathematical statement has no meaning but its symbols, regarded as physical objects, exhibit a structure that has applications. These are only restricted by the demand of consistency of the axioms of mathematics.

Hilbert: "The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose shape is immediately clear and recognizable."

Formalism

According to the formalist mathematics is not, or need not be, *about* anything, or anything beyond typographical characters and rules for manipulating them.

"In the beginning was the sign… The sign 1 is a number. A sign that begins with 1 and ends with 1, and such that in between + always follows 1 and 1 always follows + is likewise a number." (Hilbert 1922, sec. 25)

Formalism

In Cambridge I have been asked whether I believe that mathematics is about strokes of ink on paper. To this I reply that it is so in just the sense in which chess is about wooden figures. For chess does not consist in pushing wooden figures on wood.

(WWK, 141-142)

W on need for foundations

What does mathematics need a foundation for? It no more needs one, I believe, than propositions about physical objects—or about sense impressions, need an *analysis*. What mathematical propositions do stand in need of is a **clarification of their grammar**, just as do those other propositions.

The *mathematical* problems of what is called foundations are no more the foundation of mathematics for us than the painted rock is the support of a painted tower.

'But didn't the contradiction make Frege's logic useless for giving a foundation to arithmetic?' Yes, it did. But then, who said that it had to be useful for this purpose?

RFM VII §16

PI and mathematics

124. Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.

For it cannot give it any foundation either.

It leaves everything as it is.

It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other.

What right has a philosopher to talk about mathematics?

(LFM, 14)

I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from the words of our ordinary everyday language, such as ”proof”, ”number”, ”series”, “order”, etc. … All the puzzles I discuss can be exemplified by the most elementary mathematics. (LFM, 15)

The investigation is to draw your attention to facts you know quite as well as I, but which you have forgotten, or at least which are not immediately in your field of vision. **They will be quite trivial facts. I won't say anything which anyone can dispute.**

(*LFM* 22)

The development of W’s philosophy of mathematics

- TLP : Mathematics is ”a method of logic” (6.234) or ”a logical method” (6.2)
- The propositions of mathematics are equations, and therefore pseudo-propositions (6.2) i.e. similar to tautologies
- Mathematical propositions express no thoughts (6.21)
- The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics. (6.22)

What is essential about an equation is that it is not necessary to show that both expressions, which are connected by the sign of equality, have the same meaning; for this can be perceived from the two expressions themselves. (6.232)

- TLP also includes a theory of numbers as formal operations

”Middle period” (1929 to ca 1935)

- Calculus model: mathematics consists of self-contained calculi (mathematics also functions as a model for language)
- Mathematical propositions are to be understood as rules for these calculi
- An equation is ”a rule of grammar” (nMWL 161) that fixes the mening of symbols that occurrs in it
- A mathematical calculus does not need an extra-mathematical application (*PR* §109; *WWK* 105)

Late W

- Language game view (in contrast to the earlier emphasis on ”propositions”)
- Mathematics can be seen as a ”motley of techniques of proof”
- There is a *family resemblance* for instance among what we call ”number” or ”mathematics”
- ”Meaning as use”: the language games of mathematics must be seen in connection with the use of their signs (“in *mufti*”) outside mathematics, as part of an activity in “the stream of life” or as part of a “form of life”
- A mathematical ‘proposition’ functions as if it were an empirical proposition “hardened into a rule”

Number is a family resemblance concept:

—And I shall say: 'games' form a family.

And for instance the kinds of number form a family in the same way. Why do we call something a "number"? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name.

PI §67

Number is a family resemblance concept:

What kind of misunderstandings am I talking about? They arise from a tendency to assimilate to each other expressions which have very different functions in the language. **We use the word "number" in all sorts of different cases, guided by a certain analogy. We try to talk of very different things by means of the same schema.** …Hence I will have to stress the differences between things, where ordinarily similarities are stressed, though this too can lead to misunderstandings.

(LFM 14-15)

And so is *mathematics*

- Mathematics can be viewed as a "family of activities for a family of purposes" (RFM V§15) or as "ein buntes Gemisch" («motley»):

I should like to say: mathematics is a MOTLEY of techniques of proof.—And upon this is based its manifold applicability and its importance.

RFM III §46

I want to give an account of the motley of mathematics.

RFM III §48

- Mathematics can be considered a collection of various language games

…but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* of this from the changes in mathematics.) PI §23

The centrality of application in mathematics

I want to say: It is essential to mathematics that its signs are also employed in *mufti* [*im Zivil*].

It is the use outside mathematics, and so the *meaning* [*Bedeutung*] of the signs, that makes the sign-game into mathematics.

(*RFM* V, §2, 1942)

(cf. *LFM* 140–141, 169–70)

What about pure mathematics?

- It is still possible to argue that W saw "pure" symbolic mathematics as the central case of our present-day practice of mathematics: mathematical signs only get a meaning within a mathematical symbol system

It is of course clear that the mathematician, in so far as he really is 'playing a game'…[is] *acting* in accordance with certain rules. (*RFM* V, §1)

In a certain sense it is not possible to appeal to the meaning [Bedeutung] of the signs in mathematics, just because it is only mathematics that gives them their meaning.
(*RFM* V, §16)

Continuity in Ws philosophy of mathematics

- “mathematical propositions” are *not real propositions*
- “mathematical truth” is essentially *non-referential*
  - (correctness, not truth)
- we *invent* mathematical calculi and we expand mathematics by calculation and proof
  - though we learn from a proof that a theorem *can* be derived from axioms by means of certain rules in a particular way, it is *not* the case that this proof-path pre-exists our construction of it (there to be *discovered*)

What does the philosophy of mathematics deal with?

PI §254. …. What we 'are tempted to say' in such a case is, of course, not philosophy; but it is its raw material. So, for example, what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*.

What do mathematicians say about mathematics?

” I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our ’creations,’ are simply the notes of our observations.”

G. H. Hardy, *A Mathematician's Apology* (London 1941)

Main claims of Platonism in mathematics

- There is a realm of necessary facts independent of human thought (the ”world of mathematical objects”)
- Mathematical propositions describe this realm
- These facts may outrun our ability to get access to them by proofs (the existence of ”undecidable propositions” in mathematics)

Platonism represents a natural way of thinking about mathematics

for instance because
- the meaningfulness of mathematical propositions seems to demand that mathematical concepts have a reference
- mathematical proofs are compelling and yet may have conclusions which are surprising
- we seem to be able to understand some mathematical propositions without having any guarantee that proofs of them exist (cf. Goldbach's conjecture, and formerly, Fermat's theorem)
- in mathematics, we seem to be confronted with both the infinitely large and the infinitely small, which we can symbolize but not completely comprehend

- Exampels of modern day Platonists in philosophy of mathematics
  - Frege
  - Cantor
  - Hardy
  - Gödel

- W thinks Platonism is misleading or even dangerous
- Why?
  - It distorts our understanding of what mathematics and mathematical activity is
  - It gives mathematics a false inexorability and refuses to see it as a human invention and as dependent on what human beings actually do

- W wants to combat the Platonist picture that mathematics concerns "the natural history of mathematical objects" and that mathematics is a kind of natural science dealing with infinite extensions and other "mysteries of the mathematical world".
- His "anthropological perspective" on mathematics is a part of this.

Contrast this to G.H. Hardy:

"mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean."

(1941, 21)

"Fractions cannot be arranged in an order of magnitude."—First and foremost, this sounds extremely interesting and remarkable. … this proposition belongs simply and solely to mathematics, seems to concern as it were the natural history of mathematical objects themselves.

One would like to say of it e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give a warning.

When it looks as if..., we should look out.

(RFM II, §§ 39-41)

Example : Goldbach's conjecture

"Every even integer greater than 2 can be expressed as the sum of two primes."

Is this a mathematical proposition?

Mathematical propositions

Is Goldbach's theorem a mathematical proposition in the same sense as "2+2=4"?

Does this mean they are *about* a mathematical reality?

In TLP mathematical propositions are understood as similar to tautologies (i.e., pseudo-propositions)

In "middle period" W emphasizes that mathematical propositions should be understood as expressions of rules, not as descriptions

In late period, he emphasizes that mathematical propositions have a number of different uses

In all these cases, mathematical "propositions" are contrasted to ordinary propositions (that describe reality and have a truth-value)

- Hardy: "Goldbach's conjecture is a proposition, and that is why I can believe it is true". (cf. BB, 14)
- Wittgenstein:

  When Hardy says he believes Goldbach's theorem, I would ask him what his belief in this theorem led him to. What does he do? It may have led him to attempts to prove it, which shows that *some* meaning attaches to the theorem inasmuch as these activities would not have been caused by another theorem.

- However, it is misleading to consider Goldbach's conjecture to be a mathematical proposition because we do not know how to decide it, i.e., we do not know how to make it either proved (true) or refuted (false)
- "What 'mathematical questions' share with genuine questions, is simply that they can be answered."

- But how are we then to understand such unproved mathematical "propositions" as Goldbach's conjecture?
- For instance as "signposts for mathematical research, stimuli for mathematical constructions"

The function of this proposition-like structure can only very remotely be compared to the function of propositions in the ordinary sense.

In a certain sense the mathematician discovers both *question and answer*.

Russell's idea, that only the fulfilment of a wish shows us what we wished for, truly applies to mathematical wishes.

MS 163,54v-55r

Does this make mathematics arbitrary?

- In a sense (”the arbitrariness of grammar”)
- Mathematics is a human invention, a creation rather than a discovery
- But this does not undermine its certainty: ”the kind of certainty is the kind of language-game”
- Mathematical ”propositions” have a special role in our language; they are part of the ”scaffolding” of our language

Does this make mathematics arbitrary?

- Mathematical ”propostions” are norms of representation, rules, or ”standards of comparison” like the meter rod—not propositions about a mathematical reality
  - this is not a general theory of "mathematical propositions"; but at least in "certain language-games" this is a useful and illuminating distinction to make

    (cf. RFM VII, § 6)
- The agreement about these norms is **not** arbitrary

Wittgenstein’s last word?

What has to be accepted, the given, is—so one could say—*forms of life.*

Does it make sense to say that people generally agree in their judgments of colour? What would it be like for them not to?—One man would say a flower was red which another called blue, and so on.—But what right should we have to call these people's words "red" and "blue" *our* 'colour-words'?"—

How would they learn to use these words? And is the language-game which they learn still such as we call the use of 'names of colour'? There are evidently differences of degree here.

This consideration must, however, apply to mathematics too. If there were not complete agreement, then neither would human beings be learning the technique which we learn. It would be more or less different from ours up to the point of unrecognizability.

PI II p. 227

Wittgenstein's last word?

"But mathematical truth is independent of whether human beings know it or not!"

—Certainly, the propositions "Human beings believe that twice two is four" and "Twice two is four" do not mean the same. The latter is a mathematical proposition; the other, if it makes sense at all, may perhaps mean: human beings have *arrived* at the mathematical proposition. The two propositions have entirely different *uses.*

—But what would *this* mean: "Even though everybody believed that twice two was five it would still be four"?

—For what would it be like for everybody to believe that?

—Well, I could imagine, for instance, that people had a different calculus, or a technique which we should not call "calculating". But would it be *wrong*? (Is a coronation *wrong*? To beings different from ourselves it might look extremely odd.)

Of course, in one sense mathematics is a branch of knowledge,—but still it is also an *activity.* And 'false moves' can only exist as the exception. For if what we now call by that name became the rule, the game in which they were false moves would have been abrogated.

PI II, p.227-228