The Color-Exclusion Problem Revisited

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1. Introduction

It is impossible for two different colors to occur at the same place simultaneously. Therefore if a simple color-statement that ascribes a color to a spatio-temporal location is true, then any simple color-statement that ascribes another color to the same spatio-temporal location cannot be true. But Wittgenstein's Tractatus logico-philosophicus (TLP) requires that elementary propositions must be logically independent of each other[1] (I'll call this "Independence Requirement"). Thus the Independence Requirement seems to conflict with the impossibility of simultaneous presence of different colors at the same place. This color-exclusion problem has been thought to be one of the main reasons why such a simple color-statement cannot be elementary proposition of TLP.

In this paper I shall show that such a simple color-statement can be analyzed into a truth-function of elementary propositions that are logically independent of each other. This means that we can construct a system of color-descriptions which satisfies the Independence Requirement, and it will turn out that our system of color-descriptions given below reflects "the logical structure of color" (TLP 6.3751) fairly well.

Some interpreters of TLP have tried to give an analysis of color-statements that can solve the color-exclusion problem. But they don't seem to be successful. Canfield's analysis (1976, 90-93), which attempts to make the logic of color-mixture overlap with that of truth-functions (Cook also follows this line (1994, 37-39)), I think, cannot handle the color-metamerism[2] adequately. For it turns out that the Independence Requirement must be given up in so far as the metamerism is taken into account. Wedin's analysis (1992, 51-52), which follows faithfully a passage from Wittgenstein's Big Typescript (TS213, 475-476), cannot overcome the same problem as Canfield's analysis encounters. Hintikka & Hintikka offer, "as a thought-experiment", a color-ascription by "a function c which maps points in visual space into a color space" (1986, 121-124). But this proposal will encounter two difficulties : (a) it cannot satisfy the Independence Requirement ; (b) it would give rise to a contradiction that is not truth-functional but is "logical", to the contrary of TLP (6.375,5.525,4.46).

Now the multiplicity of colors can be depicted by various figures such as a line (according to the color-spectrum), a circle (e.g., Newton's color-circle), a plane figure whose shape looks like a sail of a sailboat (the chromaticity-diagram of the CIE), an octahedron (according to hue and lightness), and various solid bodies (according to hue, lightness, and saturation), &c. The analysis of color-statements in the next section, which relies on a notation which I call the "dividing notation", should be, and indeed is independent of which figure we use to depict the color-multiplicity. In order to make this clear, I shall present the notation in a general form (§2), and then apply it to a particular color-space (§3).

2. Dividing notation in its general form

As is seen below, the dividing notation relies on the binary-notation. The idea of using the binary-notation to describe colors has been suggested by Wittgenstein (Z, §368). Carruthers used this idea to construct a model of color-descriptions that satisfies the Independence Requirement (1990, 144-147)[3]. He exhibited the model whose color-space is only 1-dimensional (yet he doesn't adopt it eventually, since his interpretation of TLP isn't "phenomenalist"). I will generalize this idea into the "dividing notation" and will show that this notation is applicable to any (finite) n-dimensional space.

Suppose the color-multiplicity is depicted by a (finite) n-dimensional space $Q = Q_1 \times Q_2 \times .... \times Q_n$ adequately, a value of each $Q_m (1 \bullet m \bullet n)$ corresponds to an intensity of a piece of various features of color (e.g., hue, lightness, &c.), and it ranges over within the closed interval [0,1]. Thus the color-space Q can be defined as

$$\{ <x_1,...,x_n> \mid x_1 \in Q_1 \ \& \ .... \ \& \ x_n \in Q_n \ \& \ 0 \bullet x_1 \bullet 1 \ \& \ .... \ \& \ 0 \bullet x_n \bullet 1 \},$$

where every color corresponds to an ordered n-tuple $<x_1,...,x_n> \in Q$.

Now, for a natural number m $(1 \bullet m \bullet n)$, consider the following function D_m from Q into { 0, 1 } :

$$D_m(x_1,...,x_n) = \begin{cases} 0 & \text{if } 0 \bullet x_m \bullet 0.5 \\ 1 & \text{otherwise.} \end{cases}$$

This function assigns 0 to $<x_1,...,x_n> \in Q$ if $0 \bullet x_m \bullet 0.5$ and assigns 1 to it otherwise. This operation that assigns 0 or 1 to every n-tuple $<x_1,...,x_n> \in Q$ according to the value of its m-th member x_m just corresponds to the operation that divides the n-dimensional color-space in two with respect to the m-th dimension and assigns 0 to one section generated by the division and 1 to the other. So this function can be called a "dividing function". Thus the n-dimensional color-space Q is divided by each of these n dividing functions.

Next, for a natural number m $(1 \bullet m \bullet n)$, consider the following function D_{n+m} from Q into { 0, 1 } :

$$D_{n+m}(x_1,...,x_n) = \begin{cases} 0 & \text{if } 0 \bullet x_m \bullet 0.25 \text{ or } 0.5 < x_m \bullet 0.75 \\ 1 & \text{otherwise.} \end{cases}$$

This function assigns 0 to $<x_1,...,x_n> \in Q$ if $0 \bullet x_m \bullet 0.25$ or $0.5 < x_m \bullet 0.75$ and assigns 1 to it otherwise. So this function divides the color-space Q finer than the previous n dividing functions. If we repeat this kind of division n+n times, then we assign to every n-tuple $<x_1,...,x_n> \in Q$ a following binary-decimal up to the n+n-th place :

$$.D_1....D_nD_{n+1}....D_{n+n} \;\; (.D_1(x_1,...,x_n)....D_n(x_1,...,x_n)D_{n+1}(x_1,...,x_n)....D_{n+n}(x_1,...,x_n)).$$

We can divide the color-space Q as many times as we like. And the more times it is divided, the finer the differences of colors are articulated. If it is divided k times, then a binary-decimal up to the k-th place is assigned to each section generated by the k times of division and different decimals are assigned to different sections. Therefore if we specify a binary-decimal (up to the k-th place), we can specify a particular color that corresponds to the section to which it is assigned.

Then elementary propositions take the form

PTm

which says that

> (#) A color $<x_1,...,x_n>$ occurs at the place *P* at the time *T* , where that color corresponds to one of the sections to which a binary-decimal having '1' at the *m*-th place is assigned.

And in order to specify a color that corresponds to a section to which a binary-decimal having '0' at the m-th place is assigned, we will use the negation "¬*PTm*" of "*PTm*". So if a color $<x_1,...,x_n>$ is located in a section of Q to which the binary-decimal .01001.... is assigned, i.e., it is the case that

$$D_1(x_1,...,x_n)=0,\; D_2(x_1,...,x_n)=1,\; D_3(x_1,...,x_n)=0,\; D_4(x_1,...,x_n)=0,\; D_5(x_1,...,x_n)=1,\; ....,$$

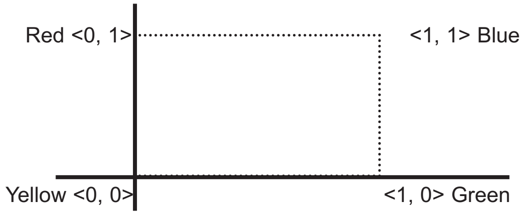
then the statement "a color $<x_1,...,x_n>$ occurs at the place *P* at the time *T* " is analyzed into the proposition "*¬PT1 & PT2 & ¬PT3 & ¬PT4 & PT5 & ....*".

The dividing notation here has been formulated in its general form. In the next section, I will apply it to a particular color-space and construct a concrete system of color-descriptions.

3. The color-exclusion problem revisited

First, assume that four colors Red, Yellow, Blue, and Green are elementary (cf., WWK, 42), and consider a color-square with its four vertexes corresponding to those four elementary colors respectively. Then regard this color-square as a coordinate-space whose origin is the vertex to which, for example, Yellow corresponds (Fig.1).

Fig.1



Thus we've got a 2-dimensional color-space Q whose *x*-axis is Yellow-Green and whose y-axis is Yellow-Red (we call the former "Q_x", the latter "Q_y". Therefore $Q = Q_x \times Q_y$). Here we have the color space :

$$\{ <x, y> \mid x \in Q_x \ \& \ y \in Q_y \ \& \ 0 \bullet x \bullet 1 \ \& \ 0 \bullet y \bullet 1 \},$$

and now every color corresponds to an ordered pair <*x, y*> ∈ Q (the distance from the point <*x, y*> to, for instance, a vertex <0, 1> depicts Red content of the color <*x, y*>).

Now I will divide this color-space Q into sixteen sections by the following four dividing functions (from Q into { 0, 1 }) :

$$D1(x, y) = \begin{cases} 0 & \text{if } 0 = x = 0.5 \\ 1 & \text{otherwise,} \end{cases}$$

$$D2(x, y) = \begin{cases} 0 & \text{if } 0 = y = 0.5 \\ 1 & \text{otherwise,} \end{cases}$$

$$D3(x, y) = \begin{cases} 0 & \text{if } 0 = x = 0.25 \text{ or } 0.5 < x = 0.75 \\ 1 & \text{otherwise,} \end{cases}$$

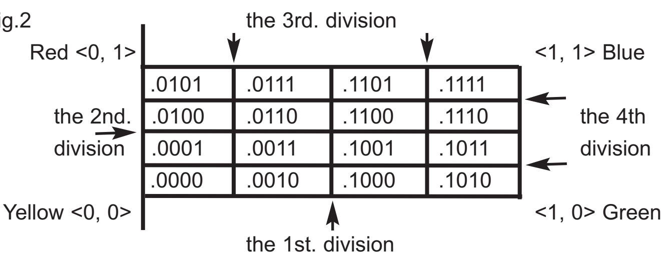
$$D4(x, y) = \begin{cases} 0 & \text{if } 0 = y = 0.25 \text{ or } 0.5 < y = 0.75 \\ 1 & \text{otherwise.} \end{cases}$$

Each of these functions assigns 0 or 1 to every <*x, y*> ∈ Q according to the value of *x* or *y*. By these functions a binary-decimal up to the *4*-th place

$$.D_1D_2D_3D_4 \ \ (.D_1(x, y)D_2(x, y)D_3(x, y)D_4(x, y))$$

is assigned to every <*x, y*> ∈ Q. In other words, every color <*x, y*> is located in a section $.D_1D_2D_3D_4$ of the 2-dimensional color-space Q. Introducing these four dividing functions amounts to fitting our color-space Q with the grid shown in the following figure.

Fig.2



What an elementary proposition "*PTm*" says is just like (#) in §2 except n = 2, and a negated elementary proposition is used in the same way as in §2. Then since the color Red corresponds to the ordered pair <0, 1> in the above figure, and it is the case that

$D_1(0, 1) = 0$, $D_2(0, 1) = 1$, $D_3(0, 1) = 0$, and $D_4(0, 1) = 1$

(i.e., Red is located in the section .0101 of Q), the statement "Red occurs at the place P at the time T " is analyzed into the proposition "¬PT1 & PT2 &¬PT3 & PT4 ". As is easily shown, elementary propositions are now independent of each other. Every (negated) elementary proposition doesn't imply any (negated) elementary proposition except itself (e.g., even if "*PT1* " is true, that doesn't imply either "*PT2* " or "¬*PT2* " is true).

And it is noteworthy that any statement that ascribes different colors to the same spatio-temporal location can now be shown to be a genuine truth-functional contradiction (cf., TLP6.3751). For example, the statement "Red and Blue occur at the place P at the time T " is analyzed into the genuine truth-functional contradiction : (¬*PT1 & PT2 &*¬*PT3 & PT4*) & (*PT1 & PT2 & PT3 & PT4*).

Moreover, from a proposition that ascribes a certain color to a certain spatio-temporal location we can deduce a negation of a proposition that ascribes another color to the same spatio-temporal location. In other words, the latter is a logical consequence of the former (under the classical two-valued propositional logic, of course). For example, the statement "Red occurs at the place *P* at the time *T*" can be analyzed into the proposition "¬*PT1 & PT2 &*¬*PT3 &PT4* ". This clearly implies "¬(¬*PT1 &*¬*PT2 &*¬*PT3 &*¬*PT4*)", which is the negation of the proposition corresponds to the statement "Yellow occurs at the place *P* at the time *T* ". —Thus we can see that our system of color-descriptions reflects "the logical structure of color" (TLP6.3751) fairly well.

4. Conclusion

The dividing notation, as we have seen, enables us to construct a system of color-descriptions that satisfies the Independence Requirement and reflects "the logical structure of color" (TLP 6.3751) fairly well. The dividing notation will be applicable not only to a system of color-descriptions but also to the systems of statements that describe sensory qualities of any kind (if its multiplicity can be mapped into a (finite) n-dimensional space[4]). Therefore we can conclude that the "completely analyzed" language of TLP (3.201,3.25) can have empirical content even with the Independence Requirement.

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Z, Zettel, ed.by G.E.M.Anscombe and G.H.von Wright, Basil Blackwell, 1967.

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Endnotes

1 TLP5.152, &c.

2 This is one of well-known color-phenomena that is characterized by the fact that there are more than one way in which one and the same color is resolved into its ingredients.

3 According to him he owes his model to Roger White (Carruthers 1990, 190, footnote10).

4 If only this condition is satisfied, the dividing notation can convert any system of statements that describe a collection of mutually exclusive properties into a system that satisfies the Independence Requirement.