Functions and Operations in the Tractatus

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Wittgenstein's early criticism of Russell's theory of types involved an insight into the limits of formal representations of mathematical reasoning that continues to find application. I will briefly review this criticism and indicate how it helped to motivate Wittgenstein's distinction in the *Tractatus* between functions and operations. I will further suggest that Wittgenstein's remark that formal concepts cannot be represented by a function has an application to formalizations of the Peano Axioms within first- and second-order logic.

Russell's type theory imposed a limit on the range of significance of a propositional function. He wrote,

> A *type* is defined as the range of significance of a propositional function, that is, as the collection of arguments for which the said function has values. Whenever an apparent variable occurs in a proposition, the range of values of the apparent variable is a type, the type being fixed by the function of which "all values" are concerned. (1908, p. 163)

This restriction on the values of a variable was introduced to avoid Russell's Paradox and some semantic paradoxes. Russell's type-theory made it impossible to speak of the totality of all objects (and hence impossible to generate Russell's Paradox) by requiring that any proposition containing an apparent variable must be of a higher type than the variable itself. Hence, no propositional function could be a possible value of its variable, and so statements about all propositions or all functions could not, except in certain specially restricted cases, be formed (1908, p. 166).

So for Russell, a propositional function determined a type which is "fixed by the function." As a limitation upon its possible arguments, the type of a function is no accidental property of that function, but rather an essential one. The type-restriction must be grasped if the function is understood, yet the restriction cannot itself be represented within the type theory, as Wittgenstein noted in 1913:

> We can never distinguish one logical type from another by attributing a property to members of the one which we deny to members of the other. Types can never

> be distinguished by saying (as is currently done) that one has these but the other has those properties, for this presupposes that there is a meaning in asserting all these properties of both types.(1961, p. 106)

Consider Russell's statement that no first-order function contains a function as an apparent variable (Russell 1908, p. 165). Attempting to substitute a function into the variable position in a first-order propositional function does not produce a false proposition. If φ is a first-order function, the expression $\varphi(\varphi(x))$ is not false but nonsense, as is its negation. Moreover, if one attempted to say that no first-order function contains a function as an apparent variable within the language of Russell's type-theory, then any statement expressing this restriction would have to contain a variable in the argument position of a first-order function which ranged over first-order functions. But the type theory prohibits just such a variable. It is, as Wittgenstein saw early on (1961, p. 108), impossible to have a theory of types.

In the Tractatus, Wittgenstein repeated his complaints with type theory (1986, 3.331-3 and 4.1241). One lesson he drew from its troubles was the need to distinguish between functions and operations:

> 3.332. No proposition can say anything about itself, because the propositional sign cannot be contained in itself (this is the "whole theory of types").
>
> 3.333. A function cannot be its own argument, because the functional sign already contains the prototype of its own argument and it cannot contain itself.

This restriction on functions is not shared by operations:

> 5.251 A function cannot be its own argument, but the result of an operation can be its own basis.

Functions and operations are distinguished in other ways also. We can express an elementary proposition as a function of its names; e.g., "fx" or "$\varphi(x,y)$" (4.24). When we do so, we concatenate the (material) function-sign and the argument-sign into a form that is specified by the logical syntax; e.g., "$\psi(\phi(fx))$" (3.333-4). The sense of the resulting proposition is expressed by the mutual spatial position of its components, given the syntax (3.1431-2). The same is not true of operations. Operations signify by indicating differences between forms of propositions rather than by combining with names to make a proposition (5.241-25). An operation manifests as a variable which shows how we can proceed from one form of proposition to another (5.24). In showing this, the variable for an operation does not generalize over a pre-given set of objects but rather exposes an internal relation between one proposition and another (5.21-2). An internal property is one that it is unthinkable that its object not possess (4.123). Operations thus bring out internal, essential properties of things without saying anything about them.

The introduction of internal relations that are shown by operations brings out a second lesson that Wittgenstein gleans from type theory. No proposition could express certain essential features of a logical type, which indicated to Wittgenstein the need to recognize special "formal concepts":

> 4.126 In the sense in which we speak of formal properties we can now speak also of formal concepts.

That anything falls under a formal concept as an object belonging to it, cannot be expressed by a proposition. But it is shown in the symbol for the object itself.

Formal concepts cannot, like proper concepts, be presented by a function.

Formal properties are internal properties or relations (4.1252). Formal concepts are expressed by a variable (4.1271). Wittgenstein indicates at 4.1252 that the number series is a formal series that is ordered by internal relations. It is expressed by the variable "[*a, x, O'x*]" (5.2522). Any formal series is generated by the repeated application of an operation to its own result (5.2521-2), and the "*x, O'x*" expression in the number series variable indicates this operation. I say "generated" because Wittgenstein thinks that formal concepts cannot be defined in terms of some pre-given collection of objects:

> 4.12721 The formal concept is already given with an object, which falls under it. One cannot, therefore, introduce both, the objects which fall under a formal concept and the concept itself, as primitive ideas. One cannot, therefore, e.g. introduce (as Russell does) the concept of a function and also special functions as primitive ideas; or the concept of number and definite numbers.

By this reasoning, the number series is not properly characterized by a function defined over a pre-given domain of entities. If Wittgenstein is correct, then the relation between a natural number and its successor must be something that is inadequately captured by, e.g., the successor relation of the Peano Axioms when this is understood as a function in the contemporary sense.

It is important to contrast Wittgenstein's conception of a function with the contemporary notion of a function, according to which functions are regarded as a species of relation, and treated in extension as mappings among sets or as sets of ordered *n*-tuples.[1] On this conception, nothing prohibits a function from taking the result of its own application as an argument, provided that the range of the function is a subset of its domain.

Wittgenstein did not regard logic in a model-theoretic way. For him, a proper logic would expose the conditions of *any* possible language by showing the logical structure of the world through tautologies (*cf.* 5.511, 6.112f.). Any restrictions necessary for avoiding paradoxes must thus appear in the logical syntax, and Wittgenstein thought his restriction on functions cut-off Russell's Paradox (3.333).

Wittgenstein's conception of a function may seem old-fashioned. Why not relax his constraints on functions and take them in extension, relying upon the Separation Axiom to remove set-theoretic paradoxes, and turning to theories of truth to resolve semantic paradoxes?

I suggest that this move obscures an element of mathematical reasoning that Wittgenstein was able to indicate with his own function/operation distinction. At 4.126 and 4.1272, Wittgenstein explicitly rejected the supposition that the signification of words like "Function" and "Number" could be presented by functions or sets. Rather, he thought that such concepts should be presented by variables which, within a logical syntax, indicate a formal property of all of their possible values (4.1271). His reason for thinking this returned to the sorts of considerations that motivated his rejection of type theory; namely, that that such concepts involve internal properties that cannot be expressed within a typed logic. A similar kind of problem arises within the context of the Peano Arithmetic when the successor function is interpreted in the standard, set-theoretic way.

On such an interpretation, the successor function is understood as a proper subset of the Cartesian product of the domain of a relational system satisfying the axioms. That this subset exists is taken as given -- the theory says nothing about how the set constituting the function is identified. Given a series of elements 0, 0', 0'', etc., the successor function s is simply given as the set consisting of <0, 0'>, <0', 0''>, etc., and if we substitute "0'" with "s(*k*)" in these pairs, the characterization of the set becomes transparently impredicative. No one doubts in practice that we know how to identify this set, but this know-how is not something that an extensional treatment of the successor function represents. Likewise, seeing that s(*k*) = *k*' presupposes apprehending the formal property had by all substitution-instances of the variable *k*. It presupposes, in terms of the *Tractatus*, a prior apprehension of the operation constituting the number series (*cf*. 4.1273). This operation is conceptually primary; it must be presupposed before a successor function can be defined in an extensional way.

Similarly, defining a function by saying, e.g., *fxyz* iff *z* = *x*+*y* presupposes the conceptually primary operation of adding numbers. We can, of course, recursively define a parallel "plus" function within the Peano Axioms, and prove inductively that it possesses certain arithmetical properties. But this is not to reduce an operation (in Wittgenstein's sense) to a function (in the contemporary sense). For although it is true that the domain and codomain of the *plus* function are independently specifiable, the *range* of *plus* is not, nor is the relation between its values and particular arguments. This relation is an internal one -- it is inconceivable that *plus* not return the values it does for its arguments. This point is completely obscured if *plus* is regarded as "just" a set of independently given entities (as it normally is).[2] So here too the extensional conception of a function captures the relevant relations among numbers only by presupposing essential, internal relations among them.

Standard first-order Peano arithmetic construes arithmetical operations in an extensional way and so misses these conceptually essential components of arithmetic. It has seemed to some philosophers that second-order logic might have an advantage here.[3] Second-order logic allows for quantification over functions and relations, and thereby apparently allows for a characterization of a series or a progression that is not possible in first-order contexts. The induction axiom, for instance, cannot be formulated in first-order logic; only an induction schema can be. Even with this schema, no set of first-order sentences true of the Peano Axioms is categorical.[4] The situation is different with second-order logic, however. The sentence:

$$\forall X(X(0) \wedge \forall x(X(x) \rightarrow X(S(x))) \rightarrow \forall x X(x))$$

states the induction axiom and, when conjoined with formulations of the other axioms, is categorical.[5] So the essential features of the arithmetical axioms and functions seem to be expressed, and categoricity is preserved.

This is not a true resolution of the difficulties noted above, however, as becomes clear when we reflect on the standard interpretation of second-order logic.[6] For here we are placed right back in the extensional, set-theoretic picture that Wittgenstein thinks fails to capture formal concepts. Formulas in standard second-order logic are interpreted in the same sorts of models as in the first-order case. For instance, where "*X*" is a predicate variable and "R" a predicate, a second-order sentence of the form "$\forall X$F" (where "F" contains "*X*" free and "R" does not occur) is defined as true in an interpretation in a purely extensional way.[7] A parallel account is given of the truth of "$\forall u$F", where "*u*" is a function variable. On such an interpretation, a second-order sentence like "$\forall u(u(x))$" treats the relation between a function and its value for an argument as if it were *itself* a function given in extension. Thus for a particular function/argument pair "*f*" and "*x*", no distinction is drawn within the logic between the formal property expressed by an operation (such that "f(*x*)" signifies an object internally related to the argument "*x*"), and a function understood as a set of ordered n-tuples. When the function is understood in this latter way, the relevant connection between the terms in the set (the elements of the ordered n-tuples) is taken for granted as soon as the set characterizing the function is assumed to be "given" through the definition of various subsets of the domain and its powersets.

No one need deny that these logics and axiom systems have enormous application. What is denied here is that they adequately characterize arithmetic without presupposing operations in Wittgenstein's sense, or something akin to them. This leads to a suggestion. The above considerations have been developed within the context of the logic of the *Tractatus*. I suggest that this fact is inessential, in that the distinctions Wittgenstein was there attempting to make between functions and operations, and between concepts that are formal and those that are not, could be made in different ways and without the apparatus of the Tractatus. For example, key aspects of the inexpressibility of formal concepts seem to have a parallel in rules, as Wittgenstein himself later examined them.[8] A rule-based analysis of these issues might have further advantages in the context of investigating functions defined intensionally.[9]

References

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Endnotes

1 Length restrictions prohibit a consideration of functions in intension here, though these are often introduced in the Lambda Calculus, Recursion Theory, and Category Theory. An investigation of intentionally characterized functions might, I suggest, be best approached from the standpoint of Wittgenstein's later reflections on rules (with which intensional functions are often identified).

2 It is also obscured if we forget that the Peano Axioms characterize arithmetic only up to isomorphism.

3 See for instance (Boolos 1975), and (Shapiro 1985).

4 By the Löwenheim-Skolem theorems, there are models of the axioms in first-order logic that are not isomorphic with one another.

5 It is categorical with the axioms ""x"y(S(x) = S(y)®x=y)" and ""x(S(x)¹0)" added.

6 Although I don't think any of the points I am about to make fundamentally change for non-standard, Henkin-type interpretations of second-order logic (which are also extensionally given).

7 Namely, it is true iff there exists a variant interpretation I' differing at most in the assignment of the suitably-valued characteristic function to R, and the result of substituting R for X in F is true in I'. See (Boolos and Jeffrey, pp. 199-200). R and the characteristic function themselves are just understood as subsets of some Cartesian product of the domain.

8 I think that the relation between a rule and its application is "internal" in the required sense. See Wittgenstein's discussion in sections 137-201 of his (1968).

9 See note 1.