# The *Tractatus* and the Need of Non-Truth-Functional Operations

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According to the *Tractatus*

$aRb$

$(\exists x) : aRx \,.\, xRb$,

$(\exists x) : aRx \,.\, xRy \,.\, yRb$, ...

is a formal series of propositions (4.1252, 4.1273) and, as such, it must be ordered by an internal relation (4.1252) and there must be an *operation* generating each term out of the preceding one (4.1273). If we take "*aRb*" as meaning that *a* is the father of *b*, then affirming the truth of one of the members of the series is equivalent to affirming that *a* is a direct ancestor of *b*. Although convinced that "the way[1] in which Frege and Russell express general propositions like the above is false" (4.1273), Wittgenstein believed he had found a method to express it in a *Begriffsschrift*.[2] In order to affirm that one of the members of a formal series of propositions is true, Wittgenstein (i) finds a way of giving a non-propositional expression to a general term of the series, (ii) treats this expression as a propositional variable, on the same level as propositional functions, and (iii) allows the operator *N* to act upon it. Let us recall the way he does this.

The first step is given at 4.1273, when Wittgenstein says "we can determine the general term of a formal series by giving its first term and the general form of the operation which generates the following term out of the preceding proposition". If we call "$O_i$" the recursive procedure used to obtain the successive terms of the series given above, then we can rewrite the series as follows:

$aRb$

$O_i{}'\,aRb$,

$O_i{}'O_i{}'\,aRb$,...

Its general term will be given by an expression in brackets –

$[aRb, x, O_i{}'x]$

– that must be taken as a propositional variable ranging over all the members of the series (5.2522). When order is irrelevant, we may use a more economical notation, adopting the convention that

$[aRb,\, x,\, O_i{}'x] = (\bar{\xi})$

and using the bar over the variable to indicate the *totality* of its values (5.501). In this case, a totality of propositions is given by a formal law of construction, and we must remember that the *Tractatus* admits of two other ways of determining a totality of propositions (5.501). By direct enumeration, we could adopt the convention that

$[p, q, r] = (\bar{\xi})$

Finally, we can adopt a propositional function whose values are to be taken as the values of $\xi$:

$[f\bar{x}] = (\bar{\xi})$

In any case, we can always apply the operation *N* to the selected group of propositions, obtaining a new proposition as a result ("$\sim p.\sim q.\sim r$" and "$\sim(\exists x).fx$", respectively). It is easy to see that in the case of $[aRb, x, O_i{}'x]$ the proposition "$NN(\bar{\xi})$" will say that at least one member of the formal series is true, i.e. that *a* is a direct ancestor of *b*, without making use of hereditary properties.

Formal series like

$aRb, O_i{}'aRb, O_i{}'O_i{}'aRb,...$

can be always associated with series of ascriptions of cardinal numbers in our language. The series above, for instance, may be seen as counting the number of interposed generations that separate a direct ancestor *a* from *b*. If we use "*fx*" as an abbreviation for the propositional function "*x* is a person in this room", then the series of propositions

There is no one in this room,

There is one person in this room,

There are two people in this room, ...

could be rewritten (using the Tractarian convention for different variables) as

$\sim(\exists x)fx$

$(\exists x)fx : \sim(\exists x,y)fx.fy$

$(\exists x,y)fx.fy :. \sim(\exists x,y,z) : fx.fy.fz$, ...

which is clearly a formal series. Let us call "$O_{ii}$" the operation leading from one term to the next in this series. If we adopt the convention that

$[\sim(\exists x)fx,\, x,\, O_{ii}{}'\,x] = (\bar{\xi})$

then "$NN(\bar{\xi})$" *could* parallel Russell's

$(\exists n).n = Nc{}'\,\hat{x}(fx)$

(i.e. "there is an *n* that is the cardinal number of the class of *f*'s") without making use of any set or type theory. I say "could" because there is an important qualification to be made here: the number of possible values of "*fx*" must be infinite. If it is not, the stock of different variables for the *x*-place in "*fx*" must be finite, and the series will have to stop at some point.[3] In this case,

$[\sim(\exists x)fx,\, x,\, O_{ii}{}'\,x]$

would be only an abbreviation for a *list* of propositions in the range of $\xi$. The same could be said of

$[aRb,\, x,\, O_i{}'\,x]$

and of any other series whose propositions were making (in the everyday language way of speaking) ascriptions of number. There is no reason indeed to stop short of quantification in general. If the values of "*fx*" are a finite totality of propositions, then "$(\exists x)fx$" will be merely an abbreviation

[1] I.e., by means of the so-called "hereditary properties" (cf. Frege 1964, §24).

[2] "Wollen wir den allgemeinen Satz: "b ist ein Nachfolger von a", in der Begriffsschrift ausdrücken… etc" (4.1273).

[3] Cf. Wittgenstein 1989, p. 155, but notice that in that context Wittgenstein is ruling out any propositional reference to an actual infinity.

for an ordinary logical sum. Of the three methods given in the *Tractatus* (5.501) for determining totalities of propositions, the second (propositional functions) and the third (formal laws of construction) would be superfluous; the first (direct enumeration) would be enough. As there is no *a priori* determination of logical space (i.e. of the totality of elementary propositions), it is not possible to determine *a priori* if there is a function with infinitely many possible values or not. If "*fx*" has only a finite number of values, then "$NN(\bar{\xi})$" (with $(\bar{\xi}) = [\sim(\exists x)fx, x, O_{ii}'x]$) will not parallel "$(\exists n).n = Nc'\hat{x}(fx)$". There will be, so to speak, a "last number" *m* to be applied to the function, and "$NN(\bar{\xi})$" will be a tautology, as can easily be seen if we imagine that "*fa*" and "*fb*" are the only possible values of "*fx*"[4]. On the other hand, if there are infinitely many values for "*fx*", then "$NN(\bar{\xi})$" is not a tautology, nor is "$N(\bar{\xi})$" a contradiction, since it is perfectly possible to imagine "*fx*" being satisfied by *all* the infinite arguments. Under the proposed interpretation, "$N(\bar{\xi})$" would be saying that there are infinitely many persons in this room.

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There is a very important characteristic of operations like $O_i$ and $O_{ii}$ that must be mentioned: they are *not* truth-operations. For suppose *a* is not the father of *b* (i.e. ~*aRb*). Must he be his grandfather? Of course not, although he can be. Again, suppose it is false to deny the existence of persons in this room – obviously it will not be necessary either to affirm or to deny the existence of exactly one. In general, we can say that no series of ascriptions of number is ordered by truth-operations. But they *are* ordered by operations, all the same.

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Ascriptions of cardinal numbers are always propositions that can be inserted in formal series like [*aRb*, *x*, $O_i'x$] and [~(∃*x*).*fx*, *x*, $O_{ii}'x$]. Numbers will not appear in these propositions, since they are substituted here by quantificational structures. From this point of view, numbers are nothing but abbreviatory devices of our languages and must disappear under analysis. Wittgenstein's thesis is that all formal properties of numbers can be derived from the generative power of the formal series they are used to translate. Arithmetical relations among numbers can always be reduced to logical relations among propositions of a formal series. My own thesis about the *Tractatus* is that Wittgenstein uses series like[*p*, *x*, *Nx*] to get an infinite model that can be applied to infinite series of any kind. The relevant properties of the "interesting" series, such as [*aRb*, *x*, $O_i'x$] and [~(∃*x*).*fx*, *x*, $O_{ii}'x$], are already present in the "dull" ones, such as [*p*, *x*, *Nx*], without the danger of a full stop. Let us see how this can be done.

Ω is defined in terms of *N* (6.01), and so in the symbol "Ω'*x*" the variable "*x*" must stand for propositions. According to 6.02, $\Omega^x$ may be taken as a new operation defined in terms of successive applications (iterations) of Ω. Therefore, if "*p*" is a proposition, "$\Omega^x{}'p$" and "$\Omega^y{}'\Omega^x{}'p$" must also be propositions for any natural numbers *x* and *y*. An immediate consequence is that a proposition with the form "$\Omega^x{}'\Omega^y{}'\Omega^z{}'p$" may also be expressed by the forms "$\Omega^x{}'\Omega^y{}'q$" (if q = $\Omega^z{}'p$) and $\Omega^x{}'r$" (if r = $\Omega^y{}'\Omega^z{}'p$). Now, if we define

$$\Omega^y{}'\Omega^x{}'p = \Omega^{x+y}{}'p \quad \text{Def.}^5$$

it will be easy to prove the equations "x+0=x" and "x+(y+1)=(x+y)+1"[6], which will give us a recursive definition of numerical addition. Proofs of the basic general properties of addition can easily be obtained by induction. We can similarly obtain proofs of the equations "x.0=0" and "x(y+1)=xy+x" assuming the definition given at 6.241:

$$(\Omega^x)^y{}'p = \Omega^{x.y}{}'p \quad Def.$$

To avoid difficulties with the use of the inverted comma in "$(\Omega^x)^y{}'p$", suffice it to extend the definition of "$\Omega^x{}'p$" to any power of the same:[7]:

$$(\Omega^x)^0{}'p = p \ Def.$$
$$(\Omega^x)^{y+1}{}'p = \Omega^x{}'(\Omega^x)^y{}'p$$

Now the proofs of "x.0=0" and "x(y+1)=xy+x" will run smoothly,[8] and we will have multiplication defined for any two natural numbers.

These are technical aspects of the Tractarian conception of number. As far as technical considerations are concerned, there is no reason to say that multiplication and addition cannot be mapped in the repeated application of *N* to propositions. But the mapping itself will not make much sense until we are able to show how it can reflect the use we make of numbers to describe reality. At this point, the reintroduction of operations which are not truth-operations is crucial.

As already shown, there can be no *a priori* grounds to affirm that $[aRb, x, O_i{}'x]$ is a formal series in the same sense as [*p, x, Nx*]. The first may come to an end, and in this case $O_i$ would lose one of the defining aspects of operations in general: the possibility of being applied to its own results. But it is exactly in propositions belonging to series like this that we are led (in everyday language) to apply numbers in the description of facts. On the other hand, a series like [*p, x, Nx*] does not involve any use of numbers. If "*p*" says "It's raining", both "$\Omega^5{}'p$" and "$\Omega^{35}{}'p$" will say "It's not raining". Numbers are not being used here to describe the weather, but to mark the particular place of a propositional sign in a formal series whose infinity is given *a priori*.

It is easy to see that, although defined in terms of the operation *N*, the series of numbers does not involve any property specifically associated with that operation. The same could be said of the definitions given for addition and multiplication. It is also easy to see that, if we define "Ω" as "*N*", we will have

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4 The three possible values of $\bar{\xi}$ would be "~*fa* . ~*fb*", "*fa* . ~*fb* .v. ~*fa* . *fb*", and "*fa* . *fb*". As "$NN\bar{\xi}$" would amount to a logical sum, the result would be a tautology.

5 Wittgenstein presupposes that the reader will supplement the recursive definition of the symbol "$\Omega^x{}'p$" at 6.02 defining addition on the model of the definition given at 6.241 for multiplication.

6

$$(i)\ \Omega^{x+0}{}'p = \Omega^0{}'\Omega^x{}'p = \Omega^x{}'p.$$
$$(ii)\ \Omega^{x+(y+1)}{}'p = \Omega^{y+1}{}'\Omega^x{}'p = \Omega'\Omega^y{}'\Omega^x{}'p = \Omega'\Omega^{x+y}{}'p = \Omega^{(x+y)+1}{}'p.$$

7 Wittgenstein presupposes that the reader will supplement the definition given at 6.241 with a definition for the symbol "$(\Omega^x)^y{}'p$" constructed on the model previously given for the symbol "$\Omega^x{}'p$"".

8

$$(i)\ \Omega^{x.0}{}'p = (\Omega^x)^0{}'p = p = \Omega^0{}'p.$$
$$(ii)\ \Omega^{x.(y+1)}{}'p = (\Omega^x)^{y+1}{}'p = \Omega^x{}'\Omega^{x.y}{}'p = \Omega^{x.y+x}{}'p.$$

$\Omega^{5}{}'p = \Omega^{35}{}'p$

but the sameness of sense, in this case, does not depend on the rules laid down for the use of numbers as exponents of operations. It only holds when the operation $\Omega$ is *N*. On the other hand,

$(\Omega^{5})^{7}{}'p = \Omega^{35}{}'p$

is an equation that can be proved inside the formal system of rules. The properties of $\Omega$ mapped in this system are all (and only) those defining a quite general recursive procedure to build propositions out of propositions. If the series $[aRb, x, O_i{}'x]$ is infinite, a procedure of this kind will be at work here. It is obvious that in this case

$O_i^{5}{}'aRb = O_i^{35}{}'aRb$

will not be true, while

will still hold. If we define the term “penteneration” as meaning the stretch between five successive generations, we can read “$(O_i^{5})^{7}{}'aRb$” as saying that there are 7 penteneration between *b* and his direct ancestor *a*. It is in contexts like this that we use arithmetical equations to calculate the number of generations mediating between *a* and *b*. Wittgenstein is trying to show that this is a practical and very useful device to abbreviate a long chain of logical deductions. The same could be said of propositions like

*There are 3 men and 2 women in this room*.

We may use the equation “3+2=5” as a rule of inference leading to the conclusion that there are 5 people in this room. But once more the *arithmetical* equation may be taken as a substitution rule, and the rule may be *justified* by the *logical* equivalence between two propositions:

$O_{ii}^{2}{}'O_{ii}^{3}{}'\sim(\exists x).fx = O_{ii}^{5}{}'\sim(\exists x).fx$

This is the Tractarian version of logicism: arithmetical equations are part of the deductive methods of logic.

* * *

Tractarian numbers are always used to *count*. This is what the identification of all ascriptions of numbers with quantifying structures amounts to. The formal series of cardinal numbers is associated, in ascriptions of number, with larger and larger groups of nested quantifiers, like

...,
(∃*x*)...,
(∃*x*,*y*)..., *etc*.

or

~(∃*x*)...,
~(∃*x*,*y*)... . (∃*x*)... ,
~(∃*x*,*y*,*z*)... . (∃*x*,*y*)..., *etc*.

In the final analysis, numbers would completely disappear. Ascriptions of number would be expressed by means of quantified propositions, and would not be able to accomplish any semantical task beyond the expressive power of quantifiers – to pick out objects and count them.

While writing the *Tractatus*, Wittgenstein was convinced that, under analysis, measuring would appear as a special case of counting. Propositions like

*This table is 3 meters long*.

would be formally analogous to

*There are 3 apples on the table*.

Both would have the same quantificational structure. Quantifiers would be counting apples in the second proposition, and meters in the first. But this cannot be done, as he acknowledges in the article about logical form. If we take measuring as a special case of counting, we cannot express the use of *standards*. We either multiply the standards or destroy the possibility of successive applications of them. When he wrote into Ramsey's copy of the *Tractatus* that “number is the fundamental idea of calculus and must be introduced as such”, he was aware that the elegant version of logicism he had presented in the *Tractatus* was a complete failure and had to be abandoned.

## References

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