A Logical Analysis of *A Series* and *B Series*

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Introduction

McTaggart's alleged proof of the unreality of time consists of the following six steps.
1. Time can be described either as an *A series* or a *B series*.
2. Time cannot exist without change.
3. There is no change without an *A series*.
4. The reality of the *A series* leads to a contradiction.
5. There is no *A series*. (Conclusion from 4)
6. Time is not real. (Conclusion from 2, 3, 5)

The controversial step in this proof is the fourth one. Dummett (1960) and Mellor (1998) supported McTaggart's proof for the fourth step, whereas many prominent *A theorists* have rejected it (cf. Oaklander and Smith (1994) for recent discussions on this issue). The following analysis will show that *A* and *B* series are, actually, closely related and that bilateral translations between both representation systems are possible. It will be also demonstrated that a consistent theory for *A series* can be constructed.

1. A Rational Reconstruction of *B* Series

According to McTaggart, time positions can be linearly ordered by a particular relation (Citation is from Poidevin and Macbeath (1993)):

> Each position is Earlier than some and Later than some of the other positions. To constitute such a series there is required a transitive asymmetrical relation, and a collection of terms such that, of any two of them, either the first is in this relation to the second, or the second is in this relation to the first. (p. 24)

Actually, this description is rather misleading; the relation should be transitive, *antisymmetrical*, and comparative. I call series of time positions characterized in this way *simple B series*.

(1) Theory for *simple B series*
B1. Transitivity: (*t1*)(*t2*)(*t3*) (*if* (*t1<t2* & *t2<t3*)*, then t1<t3*).
B2. Antisymmetry: (*t1*)(*t2*) (*if t1<t2, then not* (*t2<t1*)).
B3. Comparativity: (*t1*)(*t2*) (*t1<t2 or t1 = t2 or t2<t1*).

A *simple B series* consists of positions that are linearly ordered by relation <. In discussions about McTaggart's time theory, a *B series* is usually identified with this *simple B series*. However, McTaggart also (probably without clearly noticing it) suggested another type of *B series* which I call *complex B series*. A *complex B series* consists of not only time positions but also events that are contents of time positions.

> The series of positions which runs from earlier to later, or conversely, I shall call the *B* series. The contents of any position in time form an event. The varied simultaneous contents of a single position are, of course, a plurality of events. But, like any other substance, they form a group, and this group is a compound substance. And a compound substance consisting of simultaneous events may properly be spoken of as itself an event. (p. 24)

Here, McTaggart characterizes time positions as equivalence classes of events. As far as I know, nobody has tried to formalize this idea of McTaggart. It is one of the tasks of this paper.

(2) Core theory for *complex B series*
Theory for *simple B series*: B1&B2&B3.
B4. For any event *e* there is unique time position *t* that has *e* as a content. (*e*)(E! *t*) *content*(*e*,*t*).

In order to characterize time positions as equivalence classes of events, we need besides B4 another axiom IET that expresses the impossibility of *empty times*.

IET. For every time position *t* there is an event *e* that is a content of *t*. (*t*)(E *e*) *content*(*e*,*t*).

McTaggart seems to have accepted IET, but it is not necessary to form a *complex B series*. Hence, I treat IET only as an optional axiom and do not include it in the *theory of complex B series* (TCB).

By using relation *content*, function *time-position* can be defined. The definability of this function is guaranteed by B4. The introduction of this function enables one to define the fundamental *B-relations*, i.e. *earlier* and *simultaneous*, as well as *tense relations*, i.e. *Past*, *Now*, and *Future*.

(3) The *theory of complex B series* (TCB) consists of axioms B1-B4 and definitions BD1-BD6.
BD1. *t* is *time-position* of event *e* iff *e* is a content of *t*. (*t*)(*e*) (*t = time-position*(*e*) *iff content*(*e*,*t*)).
BD2. Event *e1* is *earlier* than event *e2* iff *time-position*(*e1*) < *time-position*(*e2*). (*e1*)(*e2*) (*earlier*(*e1*,*e2*) *iff time-position*(*e1*) <*time-position*(*e2*)).
BD3. Event *e1* is *simultaneous* with event *e2* iff *time-position*(*e1*) = *time-position*(*e2*). (*e1*)(*e2*) (*simultaneous*(*e1*,*e2*) *iff time-position*(*e1*) = *time-position*(*e2*)).
BD4. Event *e* is past at *t* iff *time-position*(*e*) < *t*. (*t*)(*e*) (*Past*(*t*,*e*) *iff time-position*(*e*) < *t*).
BD5. Event *e* is present at *t* iff *time-position*(*e*) = *t*. (*t*)(*e*) (*Now*(*t*,*e*) *iff time-position*(*e*) = *t*).
BD6. Event *e* is future at *t* iff *t* < *time-position*(*e*). (*t*)(*e*) (*Future*(*t*,*e*) *iff t* < *time-position*(*e*)).

Here, we can see that TCB uses four different types of relations, whereas the *simple B series* contains only one relation <.

Type <event, event>: *earlier*(*e1*,*e2*), *simultaneous*(*e1*,*e2*).
Type <time position, time position>: *t1<t2*.
Type <event, time position >: *content*(*e*,*t*).
Type <time position, event>: *Past*(*t*,*e*), *Now*(*t*,*e*), *Future*(*t*,*e*).

The following proposition is an immediate consequence from BD1 and BD5.

Proposition 1. Event *e* is present at *t* iff *e* is a content of *t*. (*t*)(*e*) (*Now*(*t*,*e*) *iff content*(*e*,*t*)).

Now, we can prove that TCB characterizes events properly.

Theorem 1. It follows from TCB that events are *weakly linearly ordered*, namely relations *earlier* and *simultaneous* satisfy the following conditions.

1. *simultaneous* is an equivalence relation, i.e. reflexive, symmetric, and transitive.
2. *earlier* is a partial order, i.e. irreflexive and transitive.
3. Weak Comparativity: (*e1*)(*e2*) (*earlier*(*e1*,*e2*) *or simultaneous*(*e1*,*e2*) *or earlier*(*e2*,*e1*)).

Proof. The first condition follows immediately from BD3. The second follows from B1, B2, and BD2. The third follows from B3, B4, and BD1-BD3.

It is especially important that tense relations are definable within TCB, so that statements that involve tense become expressible there. For example, we can express "at *t1*, *e1* is past, *e2* is present, and *e3* is future" by formula *Past*(*t1*,*e1*) & *Now*(*t1*,*e2*) & *Future*(*t1*,*e3*). This suggests that TCB can express *A properties.*

2. A Rational Reconstruction of *A* Series

An *A series* is a series of time positions or events that are characterized by three *tense predicates*, i.e. *past*, *present*, and *future*. McTaggart's *A series* has three essential properties. First, tense is described as a property of events and not as an operator on propositions. Second, change is expressed through change of tenses. Third, he talks about how far (in the past or in the future) an event is. I call this characterization of tense *flux-view of tense*, which McTaggart explicates by using an example of the death of Queen Anne.

"But in one respect it does change. It was once an event in the far future. It became every moment an event in the nearer future. At last it was present. Then it became past, and will always remain past, though every moment it becomes further and further past." (p.26)

Inspired by McTaggart's *A series*, Prior proposed *tense logic*. However, tense logic does not satisfy any of the three features of the *flux-view of tense*. The standard tense logic does not have the operator of presence and it cannot express the distance of an event from the present time. Contrarily, I propose to use tense predicates, *P*, *N*, and *F* for *past*, *present* and *future* respectively and to interpret *FFN*(*e*) as *more future than FN*(*e*). By using this idea, I will define the *theory of complex A series* (TCA) and show that TCA satisfies all of McTaggart's requirements for *A series*. At first, I introduce some notations.

(4) Recursive definition of CTPs (*compound tense predicates*)
The set of CTPs is the smallest set that satisfies the following conditions.
CTP1. *P*, *N*, and *F* are CTPs.
CTP2. If *X* is a CTP, then *PX*, *NX*, and *FX* are CTPs.

(5) Let *Y* be a *simple tense predicate*, i.e. *P* or *N* or *F* and *X* be a CTP.
A0. Axioms that characterize *k* as integer.
N1. (*e*) (*Y*(*0)*X*(*e*) *iff X*(*e*)).
N2. (*e*) (*N*(*1)*X*(*e*) *iff X*(*e*)).
N3. (*k*)(*e*) (*Y*(**k*+1)*X*(*e*) *iff Y*(**k*)*YX*(*e*)).
N4. (*k*)(*e*) (*F*(*-*k*)*X*(*e*) *iff P*(**k*)*X*(*e*)).
N5. Definition of *indexed tense predicate Y*[*k*]: (*k*)(*e*) (*Y*[*k*](*e*) *iff F*(**k*)*Y*(*e*)).
We call *F*(**k*)*Y*(*e*) a *CTP in F-form*.

By using (5), we can reduce any CTP to a CTP in *F-form* and then to an *indexed tense predicate*. For example, *PPNFFNNPPPFFF*(*e*) is reducible to *P*(*2)*N*(*1)*F*(*2)*N*(*2)*P*(*3)*F*(*2)*F*(*e*), i.e. F(*-1)*F*(*e*), i.e. *F*[-1](*e*). The fact that time flows in direction to the future can be described by saying "index *k* in *F*(**k*) becomes always greater". Now, TCA can be defined.

(6) The *theory of complex A series* (TCA) consists of the following axioms and definitions.
A1. Any event, sometime, becomes present. (*e*) (E! *k*) *F*(**k*)*N*(*e*).
A2. If event *e* is present, then *e* is not past. (*k*)(*e*) (*if F*(**k*)*N*(*e*), *then not* (*F*(**k*)*P*(*e*))).
A3. If event *e* is present, then *e* is past at the next moment.
(*k*)(*e*) (*if F*(**k*)*N*(*e*), *then F*(**k*+1)*P*(*e*)).
A4. If event *e* is past, then *e* remains past at the next moment.
(*k*)(*e*) (*if F*(**k*)*P*(*e*), *then F*(**k*+1)*P*(*e*)).
AD1. Future events are events that are neither present nor past.
(*k*)(*e*) (*F*(**k*)*F*(*e*) *iff* (*not*(*F*(**k*)*N*(*e*)) & *not*(*F*(**k*)*P*(*e*)))).

In order to demonstrate that TCA really describes the *flux-view of tense*, let us suppose that *d* represents the death of Queen Anne. Then, *d*'s change in the *A series* can be visualized as follows, where the right most tense shifts from future to present and from present to past:
…, *PPF*(*d*), *PF*(*d*), *N*(*d*), *FP*(*d*), *FFP*(*d*),...
The same series can be described by using *F*-form, where index *k* in *F*(**k*) becomes always greater:
…, *F*(*-2)*F*(*d*), *F*(*-1)*F*(*d*), *F*(*0)*N*(*d*), *F*(*1)*P*(*d*), *F*(*2)*P*(*d*),...
It can be also described by using *indexed tense predicates*:
…, *F*[-2](*d*), *F*[-1](*d*), *N*[0](*d*), *P*[1](*d*), *P*[2](*d*),...

Now, the *flux-view of tense* can be expressed by using *indexed tense predicates*, when we accept that every moment the index becomes greater. For the sake of simplicity, let us assume that the death of Queen Anne took place at 0. Then, McTaggart's former description can be interpreted as follows.

"It was once an event in the far future": *F*[*k*](*d*), where *k* is a large negative integer.
"It became every moment an event in the nearer future": Every moment, *k* approaches closer to 0.
"At last it was present": *N*[0](*d*).
"Then it became past": *P*[1](*d*)
"and will always remain past, though every moment it becomes further and further past": *P*[*k*](*d*), for any *k* that is greater than 0.

Next, to reject McTaggart's claim of inconsistency of *A series*, I prove the following theorem.

Theorem 2. TCA is consistent.

Proof. Let *Z* be the set of integers and *R* be the set of real numbers. Suppose *M* to be a two-sorted structure <<*R*,*Z*>, *I*> with interpretation function *I*, such that *I*(*k*) *is-in Z*, *I*(*N*) = {*a*: 0=<*a*<1, *a is-in R*}, *I*(*P*) = {*a*: *a*<0, *a is-in R*}, *I*(*F*) = {*a*: 1=<*a*, *a is-in R*}, *I*(*F*(**k*)*N*) = {*a*: *I*(*k*)=<*a*<*I*(*k*)+1, *a is-in R*}, *I*(*F*(**k*)*P*) = {*a*: *a*<*I*(*k*), *a is-in R*}, and *I*(*F*(**k*)*F*) = {*a*: *I*(*k*)+1=<*a*, *a is-in R*}. Then, you can easily examine that *M* verifies all axioms of TCA. Thus, TCA is consistent, because it has a model.

Theorem 3. TCA implies T1 and T2.
T1. If *e* is present, then *e* was future before. (*k*)(*e*) (*if F*(**k*)*N*(*e*), *then F*(**k*)*PF*(*e*)).
(i.e. (*k*)(*e*) (*if F*(**k*)*N*(*e*), *then F*(**k*-1)*F*(*e*)).

T2. If e is future, then e was future before. (k)(e) (*if* $F(*k)F(e)$, *then* $F(*k)PF(e)$).
(i.e. (k)(e) (*if* $F(*k)F(e)$, *then* $F(*k\text{-}1)F(e)$).)

Proof. Suppose $F(*k)N(e)$. Then, because of A1, *not* ($F(*k\text{-}1)N(e)$). Furthermore, because of A2 and A4, *not* ($F(*k\text{-}1)P(e)$). Thus, $F(*k\text{-}1)F(e)$ follows from AD1. Therefore, T1 holds. Next, suppose $F(*k)F(e)$. Then, because of A3 and AD1, *not* ($F(*k\text{-}1)N(e)$). Similarly, because of A4 and AD1, *not* ($F(*k\text{-}1)P(e)$). Thus, $F(*k\text{-}1)F(e)$ follows from AD1. Hence, T2 holds.

3. Relations between complex A series and complex B series

As we have seen, a *complex A series* and a *complex B series* are closely related. In this section, I will examine connections between these two. For this task, the following *bridging sentence* plays an essential role.

BS. Event e is a content of k iff e is $F(*k)N$. (k)(e) (*content*(e,k) *iff* $F(*k)N(e)$).

By using BS, we can prove that TCA is strong enough to derive TCB and that tense relations correspond to CTPs in *F*-form.

Theorem 4. TCB follows from TCA and BS.

Proof. B1, B2, and B3 follow from A0. B4 follows from A1 and BS. BD1-BD6 hold, because they are definitions.

Theorem 5. It follows from TCA and BS that tense relations correspond to CTPs, namely the following sentences hold:
TC1. (k)(e) (*Past*(k,e) *iff* $F(*k)P(e)$).
TC2. (k)(e) (*Now*(k,e) *iff* $F(*k)N(e)$).
TC3. (k)(e) (*Future*(k,e) *iff* $F(*k)F(e)$).

Proof. TC2 is obvious from BS, because (k)(e) (*Now*(k,e) *iff* k = *time-position*(e) *iff content*(e,k) *iff* $F(*k)N(e)$). From A2, A3, and A4, (k)(m)(e) (*if* $F(*k)N(e)$, *then* (*if* $k<m$, *then* $F(*m)P(e)$)). We assume, at first, $F(*k)N(e)$. Now, suppose that there is j with $j<k$ & $F(*j)P(e)$. Then, it follows from A4 that for any n with $j<n$, $F(*n)P(e)$. However, this contradicts to $F(*k)N(e)$. Thus, (k)(m)(e) (*if* $F(*k)N(e)$, *then* (*if* $F(*m)P(e)$, *then* $k<m$)). Now, we have (k)(m)(e) (*if* k = *time-position*(e), *then* ($F(*m)P(e)$ *iff* $k<m$)). This implies TC1. TC3 follows from TC1, TC2, B3, AD1, BD4-BD6, and Theorem 4.

To derive TCA from TCB and BS, we need A0 additionally, because TCA assumes that k is an integer.

Theorem 6. TCA follows from TCB, A0, BS, and TC1-TC3.

Proof. A1 follows from B4 and BS. Because of BD1 and BS, (k)(e) ($F(*k)N(e)$ *iff content*(e,k) *iff* k = *time-position*(e)). Because of TC1 and BD4, (k)(e) ($F(*k)P(e)$ *iff time-position*(e) < k). Then, A1-A4 follows from A0 and B1-B3. Furthermore, AD1 follows from B3, BD4-BD6, and TC1-TC3.

Conclusion

Our careful examination has shown that McTaggart's *A* and *B series* should be understood as *complex A* and *complex B series* respectively. I have also demonstrated how to characterize both series axiomatically and have proven consistency of *complex A series*. Furthermore, the fundamental correlations of both complex series have been explicated.

Literature

Dummett, Michael 1960 "A Defense of McTaggart's Proof of the Unreality of Time", reprinted in: Michael Dummett 1978 *Truth and other Enigmas*, London: Duckworth, 351-357.

McTaggart, John McTaggart Ellis 1927 *The Nature of Existence*, Chapter 33 "Time", Cambridge: Cambridge UP, reprinted in Poidevin and Macbeath 1993, 23-34.

Mellor, David Hugh 1998 *Real Time II*, Cambridge: Cambridge UP.

Oaklander, L. Nathan and Smith, Quentin (eds.) 1994 *The New Theory of Time*, New Haven: Yale UP.

Poidevin, Robin Le and Macbeath, Murray (eds.) 1993 *The Philosophy of Time*, Oxford: Oxford UP.

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