The Key Problems of KC

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1. The key problem of KC

According to Floyd and Putnam, we can extrapolate from Wittgenstein's 'notorius' remarks on Gödel's theorem some philosophically insightful remarks. Let "P" be the Godelian sentence for the logical system of *Principia Mathematica* (PM). Wittgenstein's key claim (KC) runs as follows:

> **KC**: If one assumes that ¬P is provable in PM, then one should give up the "translation" of P by the English sentence "P is not provable"

The key problem of KC is the following simple remark (call it KP): KC is compatible with realism. In this context, "realism" is the claim that mathematics is the study of a well–defined domain of abstract objects that exist independently of our thought, language or experience; it implies the view that arithmetic is the study of the standard model N of natural numbers.

However, this problem is considered in strict relation to the goal of providing arguments to clarify and support Wittgenstein's stance concerning Gödel's theorem. Otherwise KC would be an interesting remark. It is true that it provides genuine insight into the philosophical meaning of Gödel's theorem, but it certainly throws little light on Wittgenstein's thought. In order to understand why this is the case, it is important to follow Timothy Bays's (Bays 2006, p.6) recollection of the three uncontroversial mathematical results upon which KC is based:

> 1. If PM $\vdash \neg$P, then PM is ω-inconsistent.
>
> 2. If PM is consistent but ω-inconsistent, then all of the models of PM contain non-standard natural numbers—i.e., elements which the model treats as natural numbers but which do not correspond to any of the ordinary natural numbers.
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> 3. The translation of P as "P is not provable" depends on interpreting P at the "natural numbers alone." If we interpret P at a non-standard model—i.e., at one of the models described in 2—then there is no reason to think that this will lead to a translation of P as "P is not provable."

Bays goes on to criticize the passage between 1-3 and KC. He maintains that we shouldn't give up our translation of P as "P is not provable", in the case PM turns out to be ω-inconsistent, because "there's no reason to constrain our translation of P to the class of models which happen to satisfy PM" (Bays 2006, p.6). I agree with him on this latter point, but still think that the merit of KC is to underline that the equivalence between P and "P is not provable" holds only in the standard model. And we can think of cases in which this does matter[1]. But the issue is that the existence of non-standard model could perhaps pose a problem on choosing of how to translate P, but it is perfectly compatible with the fact that P is true in N iff P is not provable in PM and that if PM is consistent, than P is true in N (and not provable in PM).

In other words, not one of these results provides any ground for scepticism concerning the existence of the standard model N, because all the results are obtained using model-theoretic machinery, and, as has been argued by many, model theory is the realist framework *par exellence.*

There is only a way in which Floyd and Putnam's suggestion might be saved: it is possible (although questionable, see Rodych 2003) that Wittgenstein claimed something like KC; if so, he certainly had a great insight into Gödel's results. But he might have made such remarks only in order to highlight an important fact that could be acknowledged also from a realistic viewpoint. This is tantamount to claiming that KC does not help us understand what Wittgenstein's stance about Gödel's theorem was.

2. The myth of prose

Why do Foyd and Putnam think that KC is a philosophical claim of "great interest" (p. 624)? Because they believe it helps to avoid a misinterpretation of Gödel's result:

> That the Gödel theorem shows that (1) there is a well defined notion of "mathematical truth" applicable to every formula of PM; and (2) that if PM is consistent, then some "mathematical truths" in that sense are undecidable in PM, is not a mathematical result but a metaphysical claim. But that if P is provable in PM then PM is inconsistent and if ¬P is provable in PM then PM is ω-inconsistent is precisely the mathematical claim that Gödel proved. What Wittgenstein is criticizing is the philosophical naiveté involved in confusing the two, or thinking that the former follows from the latter. But not because Wittgenstein want to simply deny the metaphysical claim; rather he wants us to see how little sense we have succeeded in giving it.

That's an application to the case of Gödel's theorem of a general way of reading Wittgenstein's remarks on the Foundations of Mathematics: I will call it "the myth of prose". According to the myth of prose, the task of philosophical investigation of Mathematics is to distinguish between the real mathematical content of a theorem and some philosophical thesis often associated to it from mathematicians when they expose it informally. This apparently sensible approach leads to an implausible result. As has been argued by many, there is a perfectly legitimate mathematical sense in stating that Gödel's theorem shows that if PM is consistent, than there are sentence that are both true and undecidable in PM. Certainly, in Gödel's original paper (Gödel 1931) the theorem is formulated in syntactical terms, using the notions of consistency

1 We won't discuss this point here, but we can give a sketch of our argument. Following an hint from Martino 2006, we think this could play an important role in the formulation of Godel's second incompleteness theorem. The problem, roughly stated, is this: an idealised mathematician without any spatial or temporal limitation, could acknowledge the consistency of a system s as logical consequence of its axioms (that means: in every model – if there are – of the axioms of S, it is true that S is consistent). But not in every model for S the arithmetic sentence that should express the consistency of S (call it Con) is true. This explains why the mathematician doesn't draw it as a conclusion from the axioms of S. This could give an idea of a contest in which the translation of the predicate "Proof", involved in the construction of P and Con, as "provable" should be given up.

and ω – consistency, but in currently available semantic proofs of the theorem the notion of truth is explicitly used, thus providing a more simple and clearer demonstration than the original one (e.g. Smullyan 1992). Gödel himself proposed this concept in the introduction to his 1931 paper and, after Tarski, a precise mathematical sense can be ascribed to the notion of truth, thus leading to the central point: there is no *mathematical* reason to prefer the syntactic formulation of the Gödel's theorem to the semantic one. This does not mean that there are no reasons whatsoever: there are *philosophical* reasons, the main one being that using the notion of truth may suggest a Platonist reading of the theorem and, of course, Wittgenstein, among others, would not allow such a reading. But the problem now is to account philosophically for this rejection, and to do so, a philosophy of mathematics alternative that of Platonism would be called for. In this enterprise it is not helpful to assert that the Platonist's favourite way of stating the theorem is misleading: it is misleading only from an anti-realist view-point; this move thus merely begs the question.

I want to make a simple point: maybe Wittgenstein is really a quasi revisionist in Frascolla's sense (see Frascolla 1994), that means that he may only want to show that, without the metaphysical interpretation which it is usually accompanied by, the notion of a true but not provable proposition loses all its charm. But this is not the same as claiming that the notion of true but not provable sentence is a metaphysical one and, this is the central point: Wittgenstein must justify his position by giving philosophical reasons for it. Wittgenstein and his friend had to face the burden of the proof: the myth of prose could not help them.

The issue also becomes problematic if we contemplate that which Floyd and Putnam consider the mathematical theorem proved by Gödel:

> that if P is provable in PM then PM is inconsistent and if ¬P is provable in PM then PM is ω-inconsistent is precisely the mathematical claim that Gödel proved

Is the above, an apparently an uncontroversial mathematical result, really metaphysically neutral? I argue that it is no more neutral than the supposed "metaphysical thesis" (see Martino 2006).

What does it mean to say that a formal system is inconsistent? In textbooks on logics the usual explanation runs along the following lines:

> A System S is called inconsistent iff for some well formed formula of the language L of the system α, both α and its negation $\neg\alpha$ are theorems of the system S.

On making such a claim, we are considering the well formed formulas as a whole; we are considering all of them, and the same holds for the theorems of the system. This is tantamount to considering the well-formed formulas as a recursively enumerable set, a set isomorphic to the standard model N of the natural numbers. If there is a well defined notion of well-formed formula, as much as of theorem of a formal system or of numeral, there is a well defined notion of a structure that has the same structure as the standard model, N. Hilbert's notion of a formula as a *finite* sequences of signs is unintelligible if we do not grasp the notion of *finite*. But grasping this notion amounts to grasping the notion of natural number.

In short: if there is a well-defined notion of consistency for a formal system, there is a well-defined notion of a numeral, well-formed formula, theorem, and so forth, and there is a well-defined notion of a structure isomorphic to N. If this holds, there is a well-defined notion of mathematical truth applicable to every formula of PM, which is what we obtain when we interpret our formal language using this structure. So the supposed mathematical theorem collapses into the metaphysical thesis. The conclusion is that either the two formulations of Gödel's theorem are both metaphysical theses or they are both mathematical results: there is no room for the prose *versus* proof distinction.

Other factors make it extremely difficult to give an account of Gödel's first theorem, which avoids make reference to the model N: for example, natural numbers are used in Gödel numbering. Of course, even if we accept the semantic version of Gödel's theorem, many philosophical options alternative to Platonism are left open: we could be fictionalists, or nominalists, or intuitionists, although we could hardly be strict finitists. We might wonder whether we might be Wittgensteinians, and this issue is dealt with in the next paragraph.

3. Wittgenstein and revisionism

An important feature of Wittgenstein's philosophical reflection is his constant claim that it should not interfere with the work of mathematicians: he maintained that the clarification of the content of a mathematical theorem would never amount to giving up this very theorem. No mathematical acquisition should come under attack from philosophical analysis (the polemical target is the attempt made by intuitionists to reform classical mathematics by ruling out all non–constructive proof). This is another aspect of what I previously referred to as the myth of prose. It is acknowledged that Wittgenstein hated Set Theory and made serious efforts to contrast it, as he also did on referring to "curse of the invasion of mathematics by mathematical logic" (Wittgenstein 1956, p.19). This stance appears to contradict Wittgenstein's claim to non–revisionism. The usual reply to this objection is to state that, in discussing set theoretical topics (e.g. Cantor's diagonal proof), Wittgenstein's concern was only to make us look at them in the right way: he believed that, without all the metaphysical smoke that they are usually surrounded with, they would lose all their charm; however, this would not mean abandoning set theory as a calculus, as piece of mathematics. Herein lies the sense of Wittgenstein's claim that he didn't want not drive us out of Cantor's Paradise; he just wanted to make us realise that it is not a paradise.

It is beyond the scope of the present study to discuss whether this interpretation works for Wittgenstein's view of set theory; however, I do not believe that it works for the remarks made by Wittgenstein concerning Gödel's theorem. Although it is a controversial issue among Wittgenstein's scholars, many authoritative commentators (e.g. Rodych 2003 or Shanker 1988b) have pointed out that, in discussing Gödel's result, Wittgenstein's main concern was to show that in Mathematics the notion of truth must be identified with that of provability. This was in order to avoid a referential picture of mathematics: Wittgenstein rejected the idea that mathematics is *about* something (whether it consisted of mental, non- mental or even concrete sequences of signs is immaterial). It is not easy to see how this concept, if taken seriously, could fail to affect mathematical practice. For example, what sense could we give to a subject like model theory if we adopted Wittgenstein's picture?

Any attempt to defend Wittgenstein's claims is thus a hard job. This probably explains why so many authors have embraced the myth of prose: it saves us the trouble of doing such a job. The same advantage, as Russell said in another context, “of theft against honest toil” (Russell 1919, p. 71).

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