Marks of Mathematical Concepts

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The *Remarks on the Foundations of Mathematics* of Ludwig Wittgenstein is a “surprisingly insignificant product of a sparkling mind”, Kreisel maintained, his surprise evidently being due to a contrast he perceived between the written output posthumously published and his own conversations with Wittgenstein after 1942.[1] But evidence in the notebooks Wittgenstein kept during the first two years of the conversations Kreisel remembers reveals a not insignificant product.

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In a pocket notebook entry dated 9 March 1943 Wittgenstein wrote: “A number is, as Frege says, a property of a concept—but in mathematics it is a mark of a mathematical concept. $\aleph_0$ is a *mark* of the concept of natural number; and the *property* of a technique. $2^{\aleph_0}$ is a mark of the concept of an infinite decimal, but what is this number a property of? That is to say: of what kind of concept can one assert it empirically?”[2] The text can be found in the second edition of the *Remarks on the Foundations of Mathematics,* part VII, section 42, paragraph 5.

The pocket notebook entry is typical of the to and fro that characterizes Wittgenstein's later philosophy. The words 'mark' and 'property', traditional terms in the philosophy of mathematics, are italicized in the original. A remarkable feature of this passage is the issue of empirical assertions about the continuum, a concept marked by $2^{\aleph_0}$. After all, the real numbers are incommensurable with the natural numbers.

That makes startling the opening of the pocket notebook entry, since Frege is noted for his anti-empiricism. The opening is even more startling for what it says about Frege's definition of number. Wittgenstein's topic is number. He mentions Frege, so the reader anticipates objects because of Frege's thesis that numbers are objects.[3] But Wittgenstein does not turn to objects but instead to properties of concepts, while for Frege a number would be a second-level concept, if it were a property.

Of course, in regard to such properties Wittgenstein had once said: "Relations and properties, etc., are objects too".[4] He wrote that in a notebook on 16 June 1915; however, he would later come to criticize that view. It does not seem likely that Wittgenstein was confused about Frege's theory, given Peter Geach's report. Geach reports that Wittgenstein said, "The last time I saw Frege, as we were waiting at the station for my train, I said to him 'Don't you ever find *any* difficulty in your theory that numbers are objects?' He replied 'Sometimes I *seem* to see a difficulty—but then again I *don't* see it'.[5] That suggests that Wittgenstein was critical of Frege's theory.

In fine, Wittgenstein begins with a puzzle about the definition of number. The subject matter is not ideas or objects but the formation of the concept, which Rush Rhees emphasizes. Mathematical advances, and proofs in particular, modify concepts, as Crispin Wright says.[6] That modification should be open to empirical study. That is the point of the technical term 'technique', which is a method of a language game. I am taking proof to be the method of mathematics.

Of course, one's idea of infinite divisibility is not itself infinitely divisible, nor does one have access over the course of life to each of the natural numbers in turn, so the use of 'empirical' here will not solve the traditional problem of the nature of the continuum or of the natural numbers. Wittengenstein focuses rather on the multiplicity of concepts all captured by the concept of number than on justification. He asks of what kind of concept one can assert $2^{\aleph_0}$ empirically. In what follows I want to suggest some possible answers. But I first admit that Wittgenstein's question could also be taken rhetorically.

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Paul Bernays writes in 1935 that “it is not absolutely indubitable that the domain of complete evidence extends to all of intuitionism”, pointing out that for “very large numbers, the operations required by the recursive method of constructing numbers can cease to have a concrete meaning”.[7] Numbers produced by the operation of exponentiation “are far larger than any occurring in experience, e.g., $67^{257^{729}}$”. Georg Kreisel calls a position like that described by Bernays “strict finitism”. In their reviews of the *Remarks on the Foundations of Mathematics,* Bernays and Kreisel attribute this position to Wittgenstein, although Bernays discusses a Kantian tendency he observes in Wittgenstein's later philosophy, as well.

Hao Wang follows Bernays.[8] He adumbrates a bifurcation in foundational methods, the constructive over and against the nonconstructive, then correlates this with the difference between potential infinity and actual infinity. Wang distinguishes five foundational approaches. In doing so he restricts the “finitism in the narrower sense” specified by Bernays to anthropologism, a reduction to processes that are “feasible”. Those five foundational domains in order of increasing inclusivity are: anthropologism, finitism, intuitionism, predicativism, and platonism. Hao Wang points out that the “prevalent mood nowadays is not to choose a life mate from among the five ‘schools' but to treat them as useful reports about a same grand structure which can help us to construct a whole picture that would be more adequate than each taken alone”.

Anthropologism is for Wang the investigation of theoretical possibilities for human activity, what a person “can” do. A proof is “that which one can actually grasp”, as Kreisel says.[9] Wang offers the example that with only the stroke notation it becomes difficult to manipulate numbers

[1] Wittgenstein (1978), Kreisel (1958, sec. 13).
[2] MS 127, 69f.
[3] Parsons (1983, secs. 1-5, 9), v. Frege (1953, pp. 71-81, 116f.).
[4] Wittgenstein (1979), v. Hintikka and Hintikka (1986, ch. 2).
[5] V. Wright (1983, p. xii).
[6] Wright (1980, ch. 3).
[7] Bernays (1935, p. 265) and (1959)
[8] Wang (1958, pp. 473ff.).
[9] Kreisel (1958, sec. 7).

larger than ten. He makes use of anthropologism to explain some of Wittgenstein's most "cryptic" remarks.[10] Wang says in a passage reminiscent of a remark Wittgenstein made in a 1939 lecture on the foundations of mathematics: "if we reflect on the human elements involved, it is doubtful that a contradiction can lead to a bridge collapsing". The traditional interpretation of Wittgenstein as a strict finitist emanates from the readings of Bernays, Kreisel, Wang, and Dummett in 1958 and 1959 and is refined by Crispin Wright in 1980 and 1982.[11]

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In Kirchberg am Wechsel in the summer of 1992 Mathieu Marion spoke on the "dark cellar of platonism". Then in 1995 he first published his striking finitist interpretation of the later Wittgenstein, following three years later with a trenchant book on the foundations of mathematics. Marion says that overall the later Wittgenstein is a finitist, thereby posing a challenge to the restrictive traditional interpretation of Wittgenstein's later philosophy of mathematics as being anthropological or strictly finitistic. Two additional influences Marion mentions are Michael Wrigley and Jaakko Hintikka.

At the extreme are narrow conditions for mathematical proof, a radical antirealism that requires "producing" the proof. Marion last year published a radical antirealist reading of the *Remarks on the Foundations of Mathematics.*[12] But in earlier works Marion broadens the traditional reading by drawing in particular on the texts Wittgenstein wrote during his transitional middle period. His position is that Wittgenstein is a finitist, not a strict finitist. In a footnote Mathieu Marion compares Wittgenstein's purported finitism to the case William Tait describes.[13]

The account of finitism by Tait gives a sense to proofs of propositions quantifying over the natural numbers without assuming the axiom of infinity, roughly Russell and Ramsey's view of the status of elementary arithmetic in the *Tractatus Logico-Philosophicus.*[14] Tait depicts finitism as primitive recursive arithmetic: finitism is based on the finite sequence, thereby fulfilling what Hilbert requires, that the methods be secure, without necessarily fulfilling the same intuitions. Tait's minimal account of primitive recursive arithmetic is a form of platonism.

In 6.02 of the *Tractatus* Wittgenstein has something like the general form of the finite sequence: $[x, \xi, \Omega'\xi]$ presents a series of ξ having a first element x, a next element determined by the Ω operation $\Omega'x$, a next element following that determined in the same way $\Omega'\Omega'x$, and so on in that fashion until the final element of the series is reached. That is a diaphanous sense in which $\aleph_0$ is the property of the method of a language game. Dedekind himself had used a finite sequence construction to elucidate the meaning of the natural numbers.

The problems Wittgenstein identifies with the axiom of infinity are related to the assumption that $\aleph_0$ is part of logic. $\aleph_0$ is for Wittgenstein a property of an operation. An operation is not itself a concept of logic for Wittgenstein, since its existence is not established by its essence alone. Under these circumstances the second sentence of our main quotation suggests that succession is not in doubt empirically at any stage; that, however, is unlike strict finitism. Actually, one can take the development of Wittgenstein's thought from the *Tractatus* as a response to the (what were for Wittgenstein) unexpectedly realistic interpretations of Russell and Ramsey.

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Marion maintains that for Wittgenstein some of the real numbers are unreal, to vary Chaitin's phrase from his talk here two years ago.[15] But Wittgenstein does contemplate the differences of order Cantor defines. The difference between $\aleph_0$ and $2^{\aleph_0}$ is a difference of higher order, he says. In the third sentence of the main quotation Wittgenstein addresses the number of the continuum $2^{\aleph_0}$, which marks the concept of real number.

According to Tait's elucidatory review of Saul Kripke's book, Wittgenstein avoids the skeptical paradoxes by identifying and clarifying distinctions, not by capitulating and then taking up the strict finitism, in our case, as a default. If Wittgenstein does not concede the skeptical argument, the alleged motivation that drives the adoption of strict finitism is lost. Tait identifies four key distinctions made by Wittgenstein: understanding an expression, the meaning of an expression, my idea of the meaning, and the warrant for the expression. The skeptical paradox of sections 198–201 of the *Philosophical Investigations,* that each new step in a numeric series can be made out to accord with a rule no matter what number actually occurs, collapses these distinctions. To avoid the paradox the slogan is: interpretations do not determine meanings.

Wittgenstein considers the constructive nature of diagonal proof not only in the work we are considering but in other manuscripts written during the 1930's, as well. He uses his old operation symbols from the *Tractatus* in some of these same places. Wittgenstein did not formalize his discussion of diagonalization, but he gives a fairly accurate description. Kreisel's main objection in his ninth section is that Wittgenstein does not state that there are denumerable models of set theoretic realities that cannot be enumerated. But Wittgenstein is forcing a dialectical attack on multiple fronts.

To return to the end of the main quotation: the conception of Frege is accurate for many local cases. How far can one go with it empirically? Not too far according to strict finitism, but $2^{\aleph_0}$ marks the concept of real number. Of what is it a property? To put it plainly, many would take Wittgenstein's last question in the quotation rhetorically, but instead of not answering it, or conceding that it cannot be answered, one can provide multiple answers: $2^{\aleph_0}$ is a property of diagonalization, also a property of taking segments on a ray in Euclidean space or sets of initial segments, and so forth. In those cases an infinite number need not be "a property of a *property*". "Because we would not know what has that property. Yet Frege's definition has made an enormous amount clear".[16]

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Burton Dreben says that the anthropologistic reading is not dialectical enough, an interpretation I associate with Mi-

[10] Wang (1962, pp. 38, 40f.).
[11] Dummett (1959), and Wright (1980) and (1982).
[12] Marion (2008, 4th para.) and (2003); cf. Wittgenstein (1978, III.1 and III.5).
[13] (1998, p. 99 n.), also (1995).
[14] Tait (1981, sec. 1, 5, 13, 2, 4, 14), also (1986, sec. 1, 7th para.). Tait himself is not a finitist.
[15] Marion (1995, p. 163); cf. Wittgenstein (1978, II.34f.). For the diagonal proof that successively produces the digits of a new real number v. Cantor (1874) and Kanamori (1996, sec. 1.1).
[16] Wittgenstein (1976, p. 168).

chael Wrigley and Juliet Floyd, as well.[17] By that Dreben means that Wittgenstein's presentation is the *via negativa*: no explanations are permitted. Wittgenstein has no perfect counterbalance, no ground nor core account, not even an ideal frame. He had left that last option behind in the *Tractatus.*

But again one does not need to begin in that stance to see that there are significant internal problems with the other alternatives, especially when it comes to passages by Wittgenstein like the one about marks of mathematical concepts. For that pocket notebook entry need not be taken as containing a denial of the real numbers. So, on balance Ludwig Wittgenstein can be cleared of the charges of strict finitism and finitism.[*]

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[17] V. Parsons (1998, p. 21 nn. 41f.), Wrigley (1977), Floyd (1991).

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