Wittgenstein versus Hilbert

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The basic problem with the meta-mathematical tradition is that it has never investigated the forms of speech that enable one to talk about the *content* of sentences, even though it is well known that Frege wanted Hilbert's formulae to express thoughts (see, e.g. McGuinness 1980, p48). Frege's 'horizontal', which was a kind of content stroke, has not been copied in later developments of his logic, and only more recently has any other comparable symbol been promoted, for example the angle brackets for propositions in (Horwich 1998).

The abandonment of Fregean thoughts was not helped by the attack on propositions by Quine. But Hilbert's meta-mathematical programme was more fundamental, since that held, first of all, that it was all and only axiomatic structures that were the proper subject of foundational study. It is this study that has had the widest impact, not only in Mathematics but also in Logic. As Hilbert developed it, though, it had a singular difference in character from earlier studies of axiomatic structures. For Hilbert's approach was explicitly *meta-linguistic*, i.e. concerned just with the language, and formulae that appeared in the axioms.

Hilbert established the plausibility of his line of research with his axiomatisation of Geometry, which dispensed with Euclidean figures, and proceeded entirely by means of logic from completely explicit geometrical postulates. The removal of diagrams took foundational studies away from 'intuition' in the philosophical sense. More plainly, it takes one away from what the language in the axioms is about. As a result, despite wanting to say he had provided a foundation for 'Geometry' Hilbert had nothing to say about the lines and points in Euclid. Certainly the words 'line' and 'point' appear in Hilbert's axioms, but they were taken to apply simply to anything that satisfied the axioms. So the fact that those axioms *did* apply to Euclid's elements was quite incidental to Hilbert's interests, and remained something Hilbert did not attempt to provide any foundation for (see, e.g. McGuinness 1980, pp40-41).

The basic error in Hilbert's programme was therefore that it gave no account *at all* of what is true in a model of some formulae, being deliberately concerned entirely with the formulae themselves. Hilbert considered the consistency of his formal systems to be very important, but it would require more than consistency to establish results about numbers from proofs in the kind of arithmetical formal system he was concerned with. There would need to be some proof of the *soundness* of that formal system, on the standard interpretation, before one could show even that 2+3=5, for instance. This follows from the character of Hilbert's Meta-mathematics just in itself, it is important to note: there was no need to wait on Gödel, for instance, to point it out. Gödel's Theorems do not show, that is to say, that while some standard arithmetical truths are provable meta-mathematically, others are not. In fact *none* are, since any derivation within a formal arithmetical system must be supplemented with a demonstration of its soundness, on the standard interpretation, before any standard arithmetical facts can be proved on its basis.

A grammatical point about the difference between sentences and propositions is crucial to seeing the detail of the needed correction to Hilbert. Sentences are mentioned using quotes, but when used (on the standard interpretation) they express propositions which are designated by the associated 'that'-clauses (c.f. McGuinness 1980, p164). The turnstile symbol in systems of formal logic and arithmetic is therefore mistakenly read, if it is read 'it is deducible that'. For the turnstile symbol is a meta-level predicate of sentences, whereas the reading then given involves an object-level operator. The contrast can be made even more sharp once one remembers the fact that 'it is deducible that p' is equivalent to 'that p is deducible'. For the latter is of a subject-predicate form, and so the predicate 'is deducible' in it is an object-level predicate expressing a property of propositions. To get from the meta-level predicate of sentences to the object-level property of propositions one needs a proof of the soundness of the formal system involved, on the standard interpretation, and the processes involved in this additional matter of soundness have to be of quite a different character from any involved in the system in question. Indeed they cannot be formalistic at all.

Unlike a proposition about a sentence, a proposition about a proposition is not about a purely syntactic form, i.e. some symbols independently of their meaning. But propositions about sentences have dominated the Philosophy of Mathematics since Hilbert's Meta-mathematics got its grip. So, clearly they have done so illicitly, because of the above points. Certainly meta-linguistic remarks about sentences have, quite properly, entered into the theory of computing, since a computer, of course, cannot take account of the meaning of any of the symbols fed into it. But the bulk of mathematics is not a meta-linguistic study of symbols, and is instead concerned with propositions about other things: proving <2+3=5>, for example, rather than deriving '2+3=5'. Moreover, it is concerned with proving <2+3=5> *absolutely*, whereas derivations in a formal system are always relative to the axioms and rules that define that system. One might derive '2+3=5' from axioms 'A_1', …, 'A_n', using rules R_1, … R_n, but when proving <2+3=5> there is no such relativity. Of course a proof *is* involved in the formal case, since <'2+3=5' is derivable from axioms 'A_1', …, 'A_n', using rules R_1, … R_n> is proved. But what is then proved is not <2+3=5>.

The point shows that it is probably not an accident that most working mathematicians to this day (like Wittgenstein), give so little time to Gödel's Theorems. For, specifically, they are not relevant to the Foundations of Mathematics, if that is concerned, amongst other things, with the basis for what is true in the standard model of axiomatic arithmetics. Hence these results are not relevant to *Arithmetic*, as it was conceived before axiomatic studies of uninterpreted formal systems came into vogue, and, with them, non-standard models of such structures. In addition, a full proof of the fact that 2+3=5, for instance, is not available from within them. Instead it can be drawn from such illustrations as the stick figure with five lines that Wittgenstein considers (Wittgenstein 1978, p58f). Only in a practical case like that, where the numerical terms are applied (and so are used and not just mentioned), does one get beyond numerals, and other symbols, and begin to work with their meanings. For geometrical examples see

Wittgenstein's remarks on the tangram-like puzzle picture (Wittgenstein 1978 p55, and Diamond 1976 p53), those on the proof that a hand and a pentagram have the same number of vertices (Wittgenstein 1978 p53-4, Diamond 1976 pp71, 115), and those on generating a vertical column from a series of rectangles or parallelograms (Wittgenstein 1978 pp57, 58, and Diamond 1976 p128). In such geometrical cases it is particularly clear that no string of sentences in a formal proof can get to anything in the right category, and so no symbolic computer can do so either, since no such computer can give an interpretation to the symbols it processes. But the same point holds even in the arithmetical case.

The common convention of not showing quotation marks around formulae does not help people remember what it is that is 'provable' — one should really say 'derivable' — in a formal system. Only formulae are derivable, and, clearly, there is no 'proof', involving just a series of formulae, that Peano's postulates are true on the standard interpretation, for instance. For the expression 'that Peano's Postulates are true on the standard interpretation' is a noun phrase, and not a sentence, and so, *a fortiori*, it is not the last sentence of any rule-governed series of sentences. Neither, of course, can any arithmetical fact be in this position, since the noun phrase 'that 2+3=5', for instance, is equally not a sentence. So the proof of the arithmetical fact this noun phrase refers to has to be non-formal, at least at some stage, and can even proceed entirely in this way, as Wittgenstein has illustrated in cases such as those above.

The use of physical objects is one thing that is distinctive about Wittgenstein's proof that 2+3=5 using a picture of five sticks grouped into a pair and a trio at one end, while all are collected together at the other end. But what is also significant is that Wittgenstein's discussion does not go into any further details, or more complex cases such as are found in (Parsons 1979-80), for example. It is consequently utterly basic and fundamental, being concerned merely with the *foundations* of mathematics — in the proper sense of 'foundations'. Wittgenstein's discussion principally concerns the use of sticks, and the like, as *paradigms* — paradigms of countable things, for a start, and then of two things, of three things, of five things, etc. in the particular case above. Such paradigms help fix the normative criteria associated with the types of thing represented by the physical tokens in question. Books on Wittgenstein's Philosophy of Mathematics have not dwelt on these matters overmuch. But Frascolla is one commentator who addresses the required issues closely: he discusses in this connection Wittgenstein's diagrammatic proof that the fingers of a hand and the vertices of a pentacle are the same in number (Frascolla 1994, p133). The incorporation of such physical paradigms into the language, in other cases, such as The Standard Metre, and colour samples, is a well-known part of Wittgenstein's later analysis of 'simples' (Fogelin 1976, p108f, see also Baker and Hacker 1980).

It is ironic in this connection that Gödel believed in 'intuition', even though he was so much a Platonist that he believed there was another world of abstract objects accessible to a specifically mathematical intuition. For the traditional philosophical description for the particular use of ostension in diagrammatic proofs, was that it was a matter of applying one's 'intuition' — although that far more in the Kantian sense, which involved intuitions just of the spatio-temporal world, leading to synthetic *a priori* truths rather than trivially verbal, analytically *a priori* ones. What is also highly ironic is that the philosophical problems Hilbert overlooked in his meta-mathematical programme have a formal resolution in the improved predicate logic he himself introduced — the Epsilon Calculus — through its representation of Wittgensteinian simples (Slater 2007). The place of such simples in Mathematics, it then becomes clear, substantiates Wittgenstein's later, more sympathetic view of the synthetic *a priori*, the possibility of which had been ruled out in the Tractatus.

The more general moral to be drawn from this concerns the *extent* of the synthetic *a priori*. Euclid's use of diagrams in his geometrical proofs, and similar uses of physical figures in connection with the calculus, for example, were criticised on a number of grounds, particularly after the development of Analytical Geometry, by Descartes, and rigorous Analysis by Cauchy and Weierstrass. Hilbert's diagram-less presentation of traditional Geometry, which was a prelude to his promotion of Meta-mathematics, was, as we have seen, one outcome of this kind of attitude to the use of what Kant would have called 'intuition'. Nevertheless, it is clear that Meta-mathematics is still 'synthetic' in Kant's sense. For the results about symbols that have been favoured more recently in Logic, and Meta-mathematics still have an 'intuitive' basis. The point not only illuminates the more recent tradition, of course, but also reflects back on the more ancient one, where the use of diagrams was more prominent, and more trusted.

How does the synthetic *a priori* arise in Meta-mathematics? Here one may remember that computers necessarily operate on physical elements of various kinds — elements that are carefully controlled to be correct tokens of certain types of symbol. The quality control that guarantees that the tokens in question, in any particular case, do have the necessary representative properties — and so can be taken to be paradigmatic in Wittgenstein's sense — is hidden from the generality of users. But large departments in hardware producers, like IBM and Apple for instance, have to be involved in ensuring this. The processes in doing addition and multiplication by means of an electronic calculator, therefore, are in principle no different from the processes involved in doing the same sums with the aid of an abacus, for example, or with paper and pencil. Certainly one thinks of computers as being more reliable than humans at repetitive tasks, but that is only a matter of degree, and there is no doubt that some humans have developed very trustworthy skills with other physical processes.

The more general consequence is, therefore, that Hilbert's attempt to escape from Kantian 'intuition' and dispense with the synthetic *a priori* did no such thing. The very processes that Hilbert promoted generated knowledge of necessary truths by means of certain physical items — simply symbols in place of diagrams. But that undermines the principle behind the motivation for Hilbert's formal presentation of traditional Geometry. There is no difference *in principle* between a visual proof of a Euclidean geometric fact about circles, for instance, and a symbolic proof of the related, meta-mathematical fact about the word 'circle' in Hilbert's Formal Geometry. So the ultimate point is that Kant was entirely right about Mathematics being derived from the forms of human intuition.

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