The Multiple Complete Systems Conception as *Fil Conducteur* of Wittgenstein's Philosophy of Mathematics

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I

In Manuscript (MS) 183, *Bergen Edition*, p. 16-7 (from 05.06.1930) Wittgenstein refers to *Der Untergang des Abendlandes* and claims that Spengler had "*many* really important thoughts" that he himself had already thought. In the passage, Wittgenstein is referring to the idea that mathematics is not to be understood in the singular (as one body of knowledge built through historical accumulation), but should be taken as several independent developments through history. Spengler talks about the mathematics of the Greek world, mathematics of the Arab world, etc., as independent closed developments. However, Wittgenstein's interest in multiple systems was not properly historical. While Spengler talks about "a multiplicity of independent, closed in themselves developments" (Spengler 2003, 82), Wittgenstein talks about "multiple closed systems" (MS 183, p. 16-7). In Wittgenstein's conception, "closed" means "complete" and expresses the idea that a system does not *need* to be complemented. No improvement is required for a system to stand by its own. For Wittgenstein, the idea that there are missing truths or missing rules in a mathematical system is prompted by a pseudo-perspective outside systems.

An extensional conception of mathematics assumes that infinities are totalities of objects and that there are true mathematical statements about them which may not be grasped. The intensional conception, on the other hand, assumes that infinities are only possibilities expressed by rules (or operations). In the *Tractatus* Wittgenstein is, of course, already an intentionalist. There, *the* concept of number is presented as the series of numbers generated by a rule (operation). The "general form of an operation" (6.02) gives the rule of a series of which numbers are the exponents. Since numbers are not objects, in this conception, but properties of a formal series, it is nonsense to talk about all numbers. Thus, quantifiers in the *Tractatus* don't express truths about numbers at all. The talk about *the* concept of number suggests, however, that Wittgenstein had in mind one system at that time and that the idea of multiple complete systems came to his mind later.

In the Middle Period, any infinite is "the unlimited possibility of going on" (Wittgenstein 1975 §173). Possibilities are given by rules of sense, and not by true statements. 'Grammar' (rules of language in general) deals with the conditions of sense, while empirical science deals with what is and what is not true. Conditions of sense (rules of 'grammar') and conditions of truth are essentially different.

Wittgenstein expresses the idea that mathematical systems are complete in various ways. He claims that "there are no gaps in mathematics" (see MS 108, p. 20 and 22; also Wittgenstein 1993, 35), that "mathematics cannot be incomplete" (Wittgenstein 1975 §158), and that "the edifice of rules must be complete" (§154). The gap or the incompleteness would imply that there are sentences in the system that I cannot in principle understand or that there are "incomplete senses." This is not possible. For, from Wittgenstein's viewpoint in the early thirties, understanding *p* is precisely understanding *p*'s sense. Since "only the group of rules *determines* the sense of our signs" (§154; my translation), understanding *p* means understanding *p*'s rules. If *p* had an "incomplete sense", therefore, *p* would be a sign without rules. In this case, however, *p* would have no system either. But what if we discover a new rule for *p*? Say, *p* has two rules and we discover a third. This is, for Wittgenstein, the wrong description of what has taken place. The two rules of *p* determine what *p* is. With the inclusion of a third rule we have, actually, *p* (two rules) and *p** (three rules).

The rules of the system, which give sense to mathematical sentences, are the rules that present a method of proof that verify them (Wittgenstein 1975 §§149-50). Without a method of proof given in a system, a sentence is not really a sentence: it has no sense (§149). Thus, "it is impossible to pass simply by extension from one system to the other" (Wittgenstein 1993, 36) – since each system is in itself complete. This implies that "a question that has sense in the second system does not have to have sense, because of this, in the first system" (p. 36).

Wittgenstein's conception *could* have reshaped the philosophy of mathematics in the thirties, for it may be the only alternative to the extensional conception. It is clear that his conception avoids Platonism right from the beginning, for the completeness of mathematical systems implies that there are no truths to be *discovered* in mathematics. Changes in mathematics (seemingly extensions of old systems) are always inventions of new systems (Wittgenstein 1993, 35; 1975 §155). The system of integers, for instance, is "completely new" when compared to the system of natural numbers (see Wittgenstein 1993, 36). They only share a structure, which makes us believe that we have an extended the old system.[1]

In Wittgenstein's view, not only Platonists, but also Intuitionists, were captives of an extensionalist conception. Ultimately, they confuse questions of sense with questions of truth, for they fail to see that mathematics is not a set of truths, but a set of systems each of them constituted by internal rules. For Wittgenstein, the intuitionists were mistaken in two related ways. First, for thinking that laws of logic could be applied in a limited way. If one law applies, thought Wittgenstein, all apply, since they are essentially of the same kind: tautologies that minimally determine our concept of proposition. Second, when defending that there are propositions whose truth-value is not known (for instance, "truths" concerning Brouwer's *Pendelzahl*), the intuitionists were committed to the extensional view. For if we cannot *presently* determine the truth-value of certain sentences, then they at least must have sense independently of how we determine its sense. Undecidability implies extensionalism, for it "presupposes a subterranean connection … between the two sides [of an equation]" (Witt-

[1] The shared structure is the one-to-one correspondence between positive integers and natural numbers, the corresponding order (1,2… and +1, +2…) and the use of the four operations in both systems. However, +2 is not the same as 2, for plus two soldiers is completely different from 2 soldiers (see Wittgenstein 1993, 36). If we say "+2", we are immediately allowed to say "-2" as well.

genstein 1975 §174). A "subterranean connection" is, of course, something that exists independently of being grasped, something of which we may never have knowledge. Such a connection would be an unknown rule that determines the sense of a mathematical sentence (equation) *p*. Since we presumably understand *p*'s sense, such a rule is absurd (precisely because the rules determine *p*'s sense). Wittgenstein explicitly attributes an extensionalist commitment to Brouwer:

> If someone says (as Brouwer does) that in the case of (x) $f_1x = f_2x$, there is, as well as Yes and No, also the case of undecidability, then it means that "(x)…" is meant extensively and that we can talk of the case in which all x have an accidental property. In fact, however, we cannot talk at all about such case and "(x)" cannot be grasped extensionally in mathematics (MS 106, p. 129, my translation).

Thus, Intuitionism and Platonism accept (explicitly or implicitly) the mythology of unknown truths or rules "out there"; truths or rules that we may never be able to grasp. Wittgenstein's alternative to intuitionism is ingenious and simple. In order to be a consistent intentionalist, he thought, one has to abandon completely the idea of unknown truths and, at the same time, hold the view that all laws of logic come together. Thus, when there is a question, there is also a method to answer the question given inside a mathematical system: "It is not enough to say that *p* is provable, but it must mean: provable according to a determinate system" (MS 105 22-4). If there is no method of solution of a problem, then there is no problem, no undecidable question, as well. A question outside a system is not a question at all, for systems are not incomplete. If the question does not make sense, then, of course, *all* laws of logic fail to apply to it. This is precisely the point of Wittgenstein's comments on Brouwer:

> Actually, if the question about the truth or falsehood of a proposition is undecidable *a priori*, then, in this way, the question loses its sense and precisely in this way the propositions of logic lose their validity for it. (MS 106, p. 249, my translation; also Wittgenstein 1975 §173).

Thus, Wittgenstein's intensionalism grounded in the multiple complete systems conception expresses an alternative theory concerning the philosophy of mathematics in the thirties: "there are no gaps in mathematics. This *contradicts* the usual view" (Wittgenstein 1975 §158; my emphasis).

II

The multiple systems conception also underlies some of the most important changes in Wittgenstein's philosophy. The conception has some, so to speak, paradoxical consequences, which will bring Wittgenstein to a milder version of it and, finally, to its abandonment. It implies that if a sentence has no method of proof inside a system, it is not meaningful and, thus, it cannot be understood. It seems, therefore, difficult to explain that our belief that once I find the proof of *p*, I know the proof of *that* sentence (consequence 1). Moreover, this view excludes as senseless sentences for which, we usually believe, we don't have *yet* a proof (consequence 2) – Goldbach's conjecture and Fermat's equation, for instance. Another consequence of the multiple systems conception is that we cannot have two proofs for the same sentence, for it is only the proof (or method of proof) that gives sense to the sequence of words (consequence 3).

In a first moment, Wittgenstein bites the bullet and defends the "paradoxical" consequences of his theory. He defends that there is no such thing as two proofs for the same sentence (MS 108, p. 14; Wittgenstein 1975 §155); that one, in fact, does not know the meaning of a mathematical sentence if one does not have a method of proof for it (Wittgenstein 1993, 35); that, for instance, Fermat's equation was senseless (Wittgenstein 1975 §§149-50).

These consequences, however, might have brought Wittgenstein to change his philosophy later. In *Philosophical Remarks*, Wittgenstein already suggests that we might think that the value of a mathematical hypothesis (conjecture) is that "it trains … thoughts on a particular … region" (Wittgenstein 1975 §161). In the *Big Typescript*, Wittgenstein goes further. There, mathematical conjectures are "stimulus for mathematical research" (Wittgenstein 2005, 616; also WWK, p. 144). Since it seems strange that senseless constructions can have such a role, Wittgenstein is prone to ascribe them a kind of empirical content: they have a role similar to a hypothesis in physics (see Wittgenstein 2005, 616 and 619 about the hypothetical character of Fermat's equation). Even though this might explain how sentences outside systems (without a method of proof) are not simply nonsense, it aggravates the problem of *which* sentence is proved (*p* is not the same before and after a proof) – consequence 1. Since the methods of verification of a sentence that is an empirical generalization and a sentence that has the *a priori* generality of mathematics are *essentially* different (empirical evidence and proof are completely different methods of verification), as Wittgenstein says in *Big Typescript* 617, we cannot have the same sentence before and after the proof. Wittgenstein thinks that two methods of verification cannot give sense to the same sentence. It is only in the *Yellow Book* that Wittgenstein dissolves this tension. Wittgenstein claims there that we can think that the same sentence undergoes "a transition between a hypothesis and a grammatical rule" (Wittgenstein 2001, 70). He also says in his lectures of 1934-5: "It is quite possible for a proposition of experience to become a rule of grammar" (Wittgenstein 2001,160). In this case, the proof makes the empirical confirmation of the *same* sentence (hypothesis) superfluous.

The possibility of a transition between empirical and *a priori* sentences expresses an important break in Wittgenstein's philosophy. As we have seen, it was the defense of an intensionalist conception grounded in the distinction between questions of truth and questions of facts that brought him to defend the multiple complete systems conception. This strongly suggests that there is an important change in Wittgenstein's philosophy of mathematics after 1934. In fact, we can use the multiple systems conception to illustrate it. Concerning consequence 3, Wittgenstein claims later: "It would be, of course, nonsense to say that one sentence cannot have several proofs – for this is the way we say" (Wittgenstein 1999, p. 189). He is, then, clearly not defending the "paradoxical" consequences of his "complete systems conception" anymore. He is calling one of them "nonsense".

Even though a sympathy for the multiple systems conception never completely disappeared after many changes in the way it was presented, Wittgenstein had new plans for it in his Late Period. It should be considered merely as one conception amongst others. Wittgenstein, in his lectures from 1939, suggests that its new role is merely to oppose other views; as he says, to create new gas to spell old gas:

> I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are *equally* arbitrary. I will only invent a new interpretation to put side by side with an old one and say, "Here, choose, take your pick." I will only make gas to expel old gas. (my emphasis; Wittgenstein 1989, 14).

If my understanding of the shift of the role of the multiple complete systems conception is correct, we need to find and explanation for why exactly it took place, why the multiple complete systems conception is "equally arbitrary", and what is the point of its new role.*

Literature

Spengler, Oswald 2003 *Der Untergang des Abendlandes*. Deutscher Taschenbuch Verlag.

Wittgenstein, Ludwig 2005 *The Big Typescript: TS 213.* German-English Scholars' Edition. Luckardt, C.G. and Aue, A.E. (eds. and trs.). Blackwell.

Wittgenstein, Ludwig 2004 *Tractatus Logico-philosophicus*. McGuiness B.F. and Pears, D. (trs.); Routledge.

Wittgenstein, Ludwig 2001 *Wittgenstein's Lectures Cambridge, 1932-35*. From the notes of Alice Ambrose and Margaret Macdonald.Prometheus Books.

Wittgenstein, Ludwig *Wittgenstein's Nachlass. The Bergen Electronic Edition*. Oxford, 2000.

Wittgenstein, Ludwig 1999 *Bemerkungen ueber die Grundlagen der Mathematik*. Werkausgabe Band 6. Suhrkamp.

Wittgenstein, Ludwig 1993 *Wittgenstein und der Wiener Kreis*. Werkausgabe Band 3. Suhrkamp.

Wittgenstein, Ludwig 1989 *Wittgenstein's Lectures on the Foundations of Mathematics Cambridge 1939*. From the notes of Bosanquet, R.G., Malcolm, N., Rhees, R., and Smythies, Y. Diamond, C. (ed.). The University of Chicago Press.

Wittgenstein, Ludwig 1978 *Philosophical Grammar*. Kenny, A. (tr.). University of California Press.

Wittgenstein, Ludwig 1975 *Philosophical Remarks*. White, R. and Hargreaves, R. (trs.). The University of Chicago Press.

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