Wittgenstein, Turing, and the 'Finitude' of Language

Paul Livingston, Albuquerque, NM, USA

pmlivings@gmail.com

My aim in this paper is to consider the sense in which language is 'finite' for Wittgenstein, and also some of the implications of this question for Alan Turing's definition of the basic architecture of a universal computing machine. I shall argue that similar considerations about the relationship between finitude and infinity in symbolism play a decisive role in two of these thinkers' most important results, the "rule-following considerations" for Wittgenstein and the the proof of the insolubility of Hilbert's decision problem for Turing. Fortuitously, there is a recorded historical encounter between Wittgenstein and Turing, for Turing participated in Wittgenstein's "lectures" on the foundations of mathematics in Cambridge in 1939. Although my aim here is not to adduce biographical details, I think their exchange nevertheless evinces a deep and interesting problem of concern to both. We may put this problem as that of the relationship of language's finite symbolic *corpus* to (what may seem to be) the infinity of its meaning.

Wittgenstein's philosophy of mathematics has sometimes been described as a finitist; but, as I shall argue here, his actual and consistent position on the question of the finite and infinite in mathematics and language is already well expressed by a remark in his wartime *Notebooks*, written down on the eleventh of October, 1914: "Remember that the 'propositions about infinite numbers' are all represented by means of finite signs!" (Wittgenstein 1978, p. 10) The point is neither that signs cannot refer to infinite numbers nor that propositions referring to them are meaningless. It is, rather, that *even* propositions referring to infinite numbers – for instance the hierarchy of transfinite cardinals discovered by Cantor – must *have* their sense (and hence their capability to represent 'infinite quantities') by and through a finite symbolization. Thus, the problem of the meaning of the infinite is a problem of the *logic* or *grammar* of finite signs – of how, in other words, the (formal) possibilities of signification in a finite, combinatorial language can give us whatever access we can have to infinite structures and procedures.

In the 1939 lectures, Wittgenstein emphasizes that in speaking of understanding a mathematical structure, for instance a regular series of numbers or indeed the sequence of counting numbers themselves, we may speak of coming to "understand" the sequence; we may also speak of gaining a capability or mastering a 'technique.' Yet what it is to 'understand' (to "know how to," or "to be able to," continue "in the same way") is not clear. The issue is the occasion for Turing's first entrance into the discussion, in lecture number II:

> *Wittgenstein*: We have all been taught a technique of counting in Arabic numerals. We have all of us learned to count – we have learned to construct one numeral after another. Now how many numerals have you learned to write down?
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> *Turing:* Well, if I were not here, I should say $\aleph_0$.
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> *Wittgenstein*: I entirely agree, but that answer shows something.
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> There might be many answers to my question. For instance, someone might answer, "The number of numerals I have in fact written down." Or a finitist might say that one cannot learn to write down more numerals than one does in fact write down, and so might reply, 'the number of numerals which I will ever write down'. Or of course, one could reply " $\aleph_0$", as Turing did.
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> …
>
> I did not ask "How many numerals are there?" This is immensely important. I asked a question about a human being, namely, "How many numerals did you learn to write down?" Turing answered " $\aleph_0$" and I agreed. In agreeing, I meant that that is the way in which the number $\aleph_0$ is used.
>
> It does not mean that Turing has learned to write down an enormous number. $\aleph_0$ is not an enormous number. (Diamond 1976, p. 31)

Notably, Wittgenstein does not, here, *at all* deny the validity of the response that Turing initially (if guardedly) offers to the question about the capacity to write down numbers. Indeed, he distinguishes himself quite clearly from the finitist who would hold that the grammar of "can" goes no farther than that of "is," that I cannot justifiably say that my capacity includes any more than actually has occurred or will occur. In knowing how to write down Arabic numerals, a capacity we gain at an early age and maintain throughout our rational lives, we possess a capacity that is rightly described as the capacity to write down $\aleph_0$ different numbers. The attribution of this capacity is not, moreover, an answer to the "metaphysical" question of how many numbers there *are;* the question is, rather, what *we*, as human beings possessing this familiar capacity, are thereby capable *of.*

Yet how is this recognizably infinitary capacity underlain by our actual contact, in learning or communication, with a finite number of discrete signs (or sign-types) and a finite number of symbolic expressions of the rules for using them? It is not difficult to see this as the central question of the so-called "Rule-Following Considerations" of the *Philosophical Investigations*, some of which was already extant in manuscript by 1939 (see, e.g., PI 143-155; 185-240). However, we may also, I think, see this very question as *already* decisive in Turing's remarkable "On Computable Numbers, with an Application to the Entscheidungsproblem" published three years earlier, in 1936. Turing's aim is to settle the question whether there are numbers or functions that are not computable; that is, whether there are real numbers whose decimals are not "calculable by finite means" (Turing 1936, p. 58). He reaches the affirmative answer by defining a "computing machine" that works to transform given symbolic inputs, under the guidance of internal symbolic "standard descriptions", into symbolic outputs.

According to what has come to be called "Turing's thesis," (or sometimes the "Church-Turing" thesis), what it is for anything (function or number) to be calculable at all is for it to be calculable by "finite means," (here, using only a finite number of lexicographically distinct symbols and finitely many symbolically expressible rules for their inscription and transformation). Twice in the article (p. 59 and pp. 75-76), Turing justifies these restrictions by reference to the finitary nature of human cognition, either in memory or in terms of the (necessarily finite) number of possible "states of mind." Accordingly, a Turing machine can have only finitely many distinct states or operative configurations, and that its total "program" can be specified by a finite string of symbols.

These restrictions prove fruitful in the central argument of “On Computable Numbers,” to show that there are numbers and functions that are *not* computable in this sense. The first step is to show how to construct a *universal* Turing machine, that is, a machine which, when given the standard description of any particular Turing machine, will mimic its behavior by producing the same outputs. Because each standard description is captured by a *finite* string of symbols, it is possible to enumerate them and to work with the numbers (Turing calls them “description numbers”) directly (pp. 67-68). Given that we know how to construct a universal machine, we now assume for *reductio* that there is a machine, H, that will test each such description number to determine whether it is the description number of a machine that halts when given its own description number as an input. (p. 73). It does this by simulating the behavior of each machine when it is given its own description number as an input. We also know that H itself, since it always produces a decision, always halts. However, the machine H itself has a description number, K. Now we consider what happens when the hypothesized machine considers “itself,” that is evaluates whether the machine corresponding to the description number K halts. We know by hypothesis that the machine H halts; however, as Turing shows, it cannot. For in considering K, the machine enters into an unbreakable circle, calling for it to carry out its own procedure on itself endlessly. We have a contradiction, and therefore must conclude that there can be no such machine H (p. 73).

Turing’s central result is thus an application of the general metalogical procedure, first discovered by Cantor, known as “diagonalization.” This procedure underlies Cantor’s own identification of the transfinite cardinals, as well as Gödel’s two incompleteness theorems. In particular, the results of Gödel and Turing alike depend on the possibility of “numbering” symbolic strings in order to produce a *reflexive* structure that (in some sense) “says something” about itself. In that it always depends on the possibility of such enumeration, diagonalization (whatever else it may be) is *always an intervention on symbolic expressions*; that is, it depends decisively on the fact that formalizable procedures – for instance formalizable methods of proof or calculation -- are *necessarily* captured, if at all, in a *finite* combinatorial symbolic expression. In this sense, diagonalization and its results depend *essentially* on the fact that language must make use of a finite stock of symbols and a finite expression of rules in order to accomplish its powers of symbolization.

Now, it is familiar that Wittgenstein held, in general, a dim view of the purported *results* of various forms of the “diagonal procedure,” including both Cantor’s multiple infinites and the truth of Gödel’s “self-referential” sentence. Do these doubts, expressed prominently in the *Remarks on the Foundations of Mathematics*, imply that there is not a very similar concern about the relationship of finite symbolism to infinitary techniques operative in Wittgenstein’s own thought about rules and symbols? I think not, for the following reasons. In his critical remarks about the Gödel sentence as well as about Cantor’s multiple infinities, Wittgenstein emphasizes that the existence of a procedure – even one with no fixed end, like the procedure of writing down numbers in Arabic numerals – does not imply the existence of a superlative *object*, either a “huge number” or a completed list of decimal expansions that itself contains “infinitely many” members. However, Wittgenstein does not deny that there *is* such a procedure, and even that we can speak of it, with some justice, as one that shows (by giving sense to the proposition) that there is, for any enumerable set of decimal expansions, one that is not in this set. (RFM II-29). Indeed, he emphasizes the extent to which the procedure of diagonalization, as infinitary as it is, has a place, and a sense, within a human life (RFM VII – 43).

Gödel himself thought (e.g., van Atten 2006, p. 256) that diagonalization could demonstrate a *superlative* capacity of the human mind: that the existence of the Gödel sentence G shows that the human mind has access to a mathematical “truth” that no formal system such as *Principia Mathematica* can prove. However, as Gödel himself pointed out, we reach this conclusion about the system-excessive capacities of the human mind to grasp truth only through an essentially *informal* argument. Many subsequent commentators have followed Gödel in drawing this conclusion; but as Floyd and Putnam (2000) have recently argued, it is not obligatory to do so. In particular, we may agree with the negative side of Gödel’s result – there are formulable propositions of PM that are undecidable in the sense of being neither provable nor non-provable in PM, *if* PM is not ω-inconsistent – without affirming, as Gödel himself did, the mysterious capacity of human minds to grasp what is “forever” beyond the reach of formal methods. There are indeed strong indications in *RFM* (e.g. III-8) and elsewhere that this is just the interpretation that Wittgenstein favors.

Returning to Turing, the analogue is to take Turing’s result wholly negatively – that is, as showing that *there must be* infinitary procedures that are not capturable by any Turing machine (as Putnam (1991, p. 118) puts it, that “reason can go beyond whatever reason can formalize”) – without doing anything to show *what* these procedures actually are, or to guarantee our access to them. But such infinitary techniques, fixtures of human life that are not fixed, in their totality, by any finite symbolism, may be just what Wittgenstein is alluding to when, resolving the rule-following paradox of the *Philosophical Investigations*, he suggests that:

> 201. There is a way of grasping a rule which is *not* an *interpretation*, but which is shown in what we call ‘obeying the rule’ and ‘going against it’ from case to case.

And:

> 199. To understand a language means to be master of a technique.

There are, I think, two conclusions that can be drawn from this. The first is exegetical: Wittgenstein was certainly not in 1939, and probably never was, a finitist. That is, he *never* held that the finite character of language implied the non-existence or non-reality of infinite procedures. Rather, his focus is uniformly on the problem of the *grammar* of the infinite procedure: that is, just *how it is* that finite signs handled by finite beings gain the sense of infinity. This is none other than the radically posed question of the later Wittgenstein’s thought: the question of the nature of a technique or practice. And it leads to the second conclusion, which is not exegetical but philosophical: that the infinity of technique is not an extension or intensification of the finite; nor is it a superlative or transcendent object that lies “beyond” all finite procedures. The infinity of technique enters a human life, rather, at the point of what might seem at first a radical paradox: that of its capture in finite signs, the crossing of syntax and semantics wherever the infinite rule is thought and symbolized as finite.

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