A Deontic Logic with Temporal Qualification

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There are more than enough (syntactic, semantic, pragmatic) difficulties in constructing deontic logic. We suppose that at least some of them can be overcome by means of temporal qualification of the relation of deontic alternativeness and by modification of the so-called "standard" models. We use a temporal structure that is discrete, branching "to the left" and infinite in both directions. The system of tense-logic based on this structure has been developed thanks to works by J. P. Burgess. The essence of our development consists in providing the system with facilities for comparison of times of events being found on different "branches" of a set of possible courses of events.

I

There are more than enough difficulties in constructing deontic logic. The characters and the types of these difficulties—syntactic, semantic, pragmatic—are various ones. From them many relevant questions arise. For example, how we ought to qualify the status of variables? That is, whether they are thought to represent categories of human actions or to express sentences that represent states of affairs? If the latter is the case, what is it to be such a state of affairs?

We suppose that all obligations are conditional ones, though the conditions are expressed in various linguistic forms and often some obligations appear to be unconditional ones. Let us consider an example:

"Do not $\llcorner_p$forget to switch off your mobile telephones during the presentation$\lrcorner$; however if $\llcorner_p$you did$\lrcorner$ and $\llcorner_q$during the presentation a bell rang$\lrcorner$ $\llcorner_r$you ought to apologize$\lrcorner$". In symbolic form we have got: $O\neg p \;\&\; ((p \;\&\; q) \rightarrow Or)$.

At first sight it might seem that only the second obligation Or is conditional and the first one is unconditional. But rather soon[1] we find that the first obligation is supplied with a condition—"during the presentation".

1 As Wittgenstein writes (in "Diaries") a decomposed sentence speaks more than a non-decomposed one.

F. Stadler, M. Stöltzner (eds.), *Time and History. Zeit und Geschichte*.

However, it seems that the technical means for the expression of conditional obligations are rather more complicated than the ones for unconditional obligations. So, instead of a formula $\boldsymbol{O}(\boldsymbol{B}/\boldsymbol{A})$ where $\boldsymbol{A}$ is a condition for $\boldsymbol{B}$ to be obligatory we may write: $(\boldsymbol{A} \rightarrow \boldsymbol{OB})$ & $(\neg \boldsymbol{A} \rightarrow \neg \boldsymbol{OB})$. The second member of the conjunction allows us to account for the relevancy $\boldsymbol{A}$ with respect to $\boldsymbol{B}$. Otherwise we could fall into a fallacy and would formalize $\boldsymbol{O}\,(\boldsymbol{B}/\boldsymbol{A})$ as $(\boldsymbol{A} \rightarrow \boldsymbol{OB})$. Then $\boldsymbol{O}(\boldsymbol{B}/\boldsymbol{A})$ would be true whenever $\boldsymbol{A}$ is false and/or $\boldsymbol{OB}$ is true.

When constructing systems of deontic logic we use some principles which appear to be quite plausible we often encounter various paradoxical conclusions. So, for example, if we combine a collection principle, CP: $\boldsymbol{OA}\ \&\ \boldsymbol{OB} \rightarrow \boldsymbol{O}\,(\boldsymbol{A}\ \&\ \boldsymbol{B})$ with the Kantian principle ("ought" implies "can"), KP: $\boldsymbol{OA} \rightarrow \Diamond \boldsymbol{B}$ we may obtain[2]:

$(\boldsymbol{OA}\ \&\ \boldsymbol{OB}) \rightarrow (\neg\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B}) \rightarrow \Diamond(\boldsymbol{OB}\ \&\ \boldsymbol{B}))$:

1. $(\boldsymbol{OA}\ \&\ \boldsymbol{OB})$ —a premise
2. $\neg\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —a premise
3. $\boldsymbol{OA}\ \&\ \boldsymbol{OB} \rightarrow \boldsymbol{O}(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —CP
4. $\boldsymbol{O}(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —1, 3, modus ponens
5. $\boldsymbol{O}(\boldsymbol{A}\ \&\ \boldsymbol{B}) \rightarrow \Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —KP

[6.] $\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —4, 5, modus ponens

So, the lines 2 and 6 yield a contradiction.

Or let us combine two other principles that also appear to be quite plausible ones. According to the first of them, if we are obligated to provide a state of affairs $\boldsymbol{A}$ and it is the case that some other state of affairs $\boldsymbol{B}$ would be incompatible with $\boldsymbol{A}$ then we are obligated to act in such a way that there would no be $\boldsymbol{B}$: $\boldsymbol{OA}\ \&\ (\boldsymbol{B} \rightarrow \neg\boldsymbol{A}) \rightarrow \boldsymbol{O}\,\neg\boldsymbol{B}$. According to the second principle: $\boldsymbol{O}\neg\boldsymbol{B}\ \neg\boldsymbol{OB}$. Then the following formula is proved:

$(\boldsymbol{OA}\ \&\ \boldsymbol{OB}) \rightarrow (\neg\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B}) \rightarrow (\boldsymbol{OB}\ \&\ \neg\boldsymbol{OB}))$:

1. $\boldsymbol{OA}\ \&\ \boldsymbol{OB}$ —a premise
2. $\neg\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —a premise
3. $\neg\Diamond(\boldsymbol{A}\ \&\ \boldsymbol{B}) \rightarrow \Box\neg(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —a modal principle
4. $\Box\neg(\boldsymbol{A}\ \&\ \boldsymbol{B})$ —2, 3, modus ponens

2 Here we follow a principle formulated by G. Vico: "verum quod factum", i.e. "it is possible to understand only what you have made yourselves".

5.	$\neg(\boldsymbol{A} \,\&\, \boldsymbol{B})$	—4, the removal of necessity
6.	$\boldsymbol{B} \rightarrow \neg\boldsymbol{A}$	—5, propositional calculus
7.	$\boldsymbol{OA}$	—1, the removal of conjunction
8.	$\boldsymbol{OA} \,\&\, (\boldsymbol{B} \rightarrow \neg\boldsymbol{A}) \rightarrow \boldsymbol{O}\neg\boldsymbol{B}$	—the first principle
9.	$\boldsymbol{OA} \,\&\, (\boldsymbol{B} \rightarrow \neg\boldsymbol{A})$	—7, 6, the introduction of conjunction
10.	$\boldsymbol{O}\neg\boldsymbol{B}$	—9, 8, modus ponens
11.	$\boldsymbol{O}\neg\boldsymbol{B} \rightarrow \neg\boldsymbol{OB}$	—the second principle
12.	$\neg\boldsymbol{OB}$	—10, 11, modus ponens
13.	$\boldsymbol{OB}$	—1, the removal of conjunction
[14.]	$\boldsymbol{OB} \,\&\, \neg\boldsymbol{OB}$	—13, 12, the introduction of conjunction

So, the line 14 is an inconsistent formula.

II

It seems quite natural, after G.H. von Wright (1996), to accept the following definitions.

Definition 1: A state of affairs that can be produced or destroyed, prevented from coming about or from vanishing (if it is just there) is called a *doable* state.

Definition 2: A state of affairs is called *doable in pragmatical sense* if its obtaining or not-obtaining in given conditions can be a result of human action.

Definition 3:

(1) By a *genuine norm* is called such an obliging norm, ***O***-norm or a permitting norm, ***P***-norm the content of which is a doable state in the pragmatical sense.

(2) A norm the content of which is a necessary or impossible state of affairs is called *non-genuine*, or *spurious*.

Definition 4:

(1) A set of ***O***-norms is *deontically consistent* if the conjunction of their contents expressed a doable state of affairs is an obtainable formula.

(2) Every set of ***P***-norms is *deontically consistent*.

As it is known, in semantical investigations of deontic logic a so-called “standard model” appears as the following:

$$\mu = \langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{V} \rangle$$

in which $\boldsymbol{W}$ is a (non-empty) set of possible worlds, $\boldsymbol{R}$ is a binary relation of deontic alternativeness, and $\boldsymbol{V}$ is a mapping from the set of propositional letters $\boldsymbol{Var} = \{p_0, p_1, p_2, \ldots\}$ to subsets of the set $\boldsymbol{W}$. So a proposition p_i is true in a possible world α if and only if α is included into a subset w_i of $\boldsymbol{W}$ where w_i is a product of the mapping. Commonly the evaluation is extended to the use of deontic operators. For example:

$$V(\boldsymbol{O}\mathrm{A}) = \{\alpha \in W: \forall\beta \in W(\alpha R\beta \Rightarrow \beta \in V(\mathrm{A}))\}.$$

Let us note, in standard models there is *no* expression of dependence of norms on time.

III

We suppose that at least some of the difficulties in constructing deontic logic can be overcome by means of temporal qualification of the relation of deontic alternativeness and by modification of the so-called "standard" models. We use a temporal structure that is discrete, branching "to the right" and infinite in both directions. The system of tense-logic based on this structure has been developed thanks to works by J.P. Burgess (1979; 1980). The essence of our development consists in providing the system with means for the comparison of times of events being found on different "branches" of a set of possible courses of events.

The basic relation of precedence is defined as a degree of an *elementary* relation $<'$ which is characterized by the following features:

(i) irreflexivity: $\forall\mathbf{x}\neg(\mathbf{x} <' \mathbf{x})$
(ii) uniqueness of time quantum: $\forall\mathbf{x}\forall\mathbf{y}(\mathbf{x} <' \mathbf{y} \rightarrow \neg\exists\mathbf{z}(\mathbf{x} <' \mathbf{z} \,\&\, \mathbf{z} <' \mathbf{y}))$
(iii) infiniteness: $\forall\mathbf{x}\exists\forall\mathbf{y}(\mathbf{x} <' \mathbf{y}) \;\&\; \forall\mathbf{x}\exists\mathbf{y}(\mathbf{y} <' \mathbf{x})$
(iv) treelikeness: $\forall\mathbf{x}\forall\mathbf{y}\forall\mathbf{z}(\mathbf{y} <' \mathbf{x} \;\&\; \mathbf{z} <' \mathbf{x} \rightarrow \mathbf{y} = \mathbf{z})$
(v) connectedness: $\forall\mathbf{x}\forall\mathbf{y}\{\mathbf{x} \neq \mathbf{y} \rightarrow$
$\exists\mathbf{z}[\exists\mathbf{v}_1\exists\mathbf{v}_2 \ldots \exists\mathbf{v}_{n-1}(\mathbf{z} <' \mathbf{v}_1 \;\&\; \mathbf{v}_1 <' \mathbf{v}_2 \;\&\ldots\&\; \mathbf{v}_{n-1} <' \mathbf{x}) \;\&$
$\exists\mathbf{w}_1\exists\mathbf{w}_2 \ldots \exists\mathbf{w}_{m-1}(\mathbf{z} <' \mathbf{w}_1 \;\&\; \mathbf{w}_1 <' \mathbf{w}_2 \;\&\ldots\&\, \mathbf{w}_{m-1} <' \mathbf{y})]\}$

The definition of the degree of the relation $<'$ is the following:

(1) $\mathbf{x} <^1 \mathbf{y}$ if and only if (iff) $\mathbf{x} <' \mathbf{y}$
(2) $\mathbf{x} <^n \mathbf{y}$ iff $\exists \mathbf{v_1} \exists \mathbf{v_2} \ldots \exists \mathbf{v_{n-1}} (\mathbf{x} <' \mathbf{v_1} \ \& \ \mathbf{v_1} <' \mathbf{v_1} \ \& \ldots \& \mathbf{v_{n-1}} <' \mathbf{y})$

Also, let us introduce two conditions:

(n^+): There exists a relation only with a finite (though not limited by any concrete number) degree between the elements of the basic set
(n^{++}): There are no "loops": $\forall \mathbf{n}(\mathbf{x} <^n \mathbf{y} \rightarrow \mathbf{x} \neq \mathbf{y})$

Now we introduce the definition of a *complete* relation on the basic set:

$$\mathbf{x} < \mathbf{y} \text{ iff } \exists \mathbf{n} \, (\mathbf{x} <^n \mathbf{y})$$

This relation is characterized by the following features:

(i) irreflexivity: $(\mathbf{x})\neg(\mathbf{x} < \mathbf{x})$
(ii) transitivity: $\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z}(\mathbf{x} < \mathbf{y} \ \& \ \mathbf{y} < \mathbf{z} \rightarrow \mathbf{x} < \mathbf{z})$
(iii) infiniteness: $\forall \mathbf{x} \exists \mathbf{y}(\mathbf{x} < \mathbf{y}) \ \& \ \forall \mathbf{x} \exists \mathbf{y}(\mathbf{y} < \mathbf{x})$
(iv) treelikeness: $\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z}(\mathbf{y} < \mathbf{x} \ \& \ \mathbf{z} < \mathbf{x} \rightarrow (\mathbf{y} < \mathbf{z}) \vee (\mathbf{y} = \mathbf{z}) \vee (\mathbf{z} < \mathbf{y}))$
(v) connectedness: $\forall \mathbf{x} \forall \mathbf{y}(\mathbf{x} \neq \mathbf{y} \rightarrow \exists \mathbf{z}(\mathbf{z} < \mathbf{x} \ \& \ \mathbf{z} < \mathbf{y}))$
(vi) discreteness: $\forall \mathbf{x}[\exists \mathbf{y}(\mathbf{y} < \mathbf{x}) \rightarrow \exists \mathbf{y}(\mathbf{y} < \mathbf{x} \ \& \ \neg \exists \mathbf{z}(\mathbf{y} < \mathbf{z} \ \& \ \mathbf{z} < \mathbf{x}))] \ \&$
$\forall \mathbf{x}[\exists \mathbf{y}(\mathbf{x} < \mathbf{y}) \rightarrow \exists \mathbf{y}(\mathbf{x} < \mathbf{y} \ \& \ \neg \exists \mathbf{z}(\mathbf{x} < \mathbf{z} \ \& \ \mathbf{z} < \mathbf{y}))]$

IV

Thus, we will use the above-described model of time $\mathfrak{I}_b = \langle T, < \rangle$ in which $T = \{x, y, z, \ldots\}$ is a basic (non-empty) set of "moments" and < is a binary relation "earlier-later than".

Supposing that none of the "possible futures" could be qualified as a real one and since we do not want to make concessions to fatalism, we use the following temporal operators:

F′A for "It is necessary that it will be the case that ***A***";
F″A for "It is necessary that at the appointed time it will be the case that ***A***";
G′A for "It is necessary that it will always be that ***A***";
PA for "It has been the case that ***A***".

We define the common abbreviations:

$gA = \neg F'\neg A$, i.e. "it is possible that it will always be that A";
$g'A = \neg F''\neg A$, i.e. "it is possible that it will always, with exception of some time, be the case that A";
$fA = \neg G\neg A$, i.e. "it is possible that it will be the case that A";
$HA = \neg P\neg A$, i.e. "it has always been the case that A".

Our supplement to Burgess' system consists in *the axioms of discreteness* and also in axioms which circumscribe characteristics of the operator $F''A$. For the constructed system K_b proofs of its soundness, adequacy, and thus, *completeness* have been provided. There is also the proof of its *decidability* (Karavaev 1992).

V

Now for the possible worlds a relation of "*historical identity*" is introduced:

$$\alpha \cong_t \beta \text{ if and only if } \alpha(t') = \beta(t') \text{ for every } t' < t,$$

and the relation of deontic alternativeness is relativized with respect to times:

$$R_t\text{: if } \alpha R_t \beta \text{ then } \alpha \cong_t \beta$$

Now a valuation is the following:

$$V_t\text{: if } \alpha(t) = \beta(t) \text{ then } \alpha \in w_i(t) \text{ if and only if } \beta \in w_i(t).$$

Thus revised by means of the temporal qualification model $\mu_t = \langle W_t, R_t, V_t \rangle$ where W_t is a Cartesian product of the set W and the set T, the propositions are evaluated with respect to a pair $\langle \alpha, t \rangle$ and the expression $\mu_t \models A(\alpha, t)$ designates "A is true in the world α at time t".

All the conditions of truth are also relativized:

(1) $\mu_t \models p_i(\alpha, t)$ if and only if (iff) $\alpha \in w_i(t)$, for $i = 0, 1, 2, \ldots$
(2) $\mu_t \models A(\alpha, t)$ iff $\langle \alpha, t \rangle \in V_t(A)$
(3) $\mu_t \models \neg A(\alpha, t)$ iff not $\mu_t \models A(\alpha, t)$ (or $\mu_t \models A(\alpha, t)$)

(4) $\mu_t \vDash (\boldsymbol{A} \rightarrow \boldsymbol{B})(\boldsymbol{\alpha}, \boldsymbol{t})$ iff not $\mu_t \vDash \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{t})$ or $\mu_t \vDash \boldsymbol{B}(\boldsymbol{\alpha}, \boldsymbol{t})$
(5) $\mu_t \vDash \boldsymbol{OA}(\boldsymbol{\alpha}, \boldsymbol{t})$ iff $\forall \boldsymbol{\beta} \in \boldsymbol{W}_t(\alpha \boldsymbol{R}_t \boldsymbol{\beta} \Rightarrow \mu_t \vDash \boldsymbol{A}(\boldsymbol{\beta}, \boldsymbol{t}))$
(6) $\mu_t \vDash \boldsymbol{HA}(\boldsymbol{\alpha}, \boldsymbol{t})$ iff $\forall \boldsymbol{t'} \in \boldsymbol{T}(\boldsymbol{t'} < \boldsymbol{t} \Rightarrow \mu_t \vDash \boldsymbol{A}(\alpha, \boldsymbol{t'}))$
(7) $\mu_t \vDash \boldsymbol{PA}(\boldsymbol{\alpha}, \boldsymbol{t})$ iff $\exists \boldsymbol{t'} \in \boldsymbol{T}(\boldsymbol{t'} < \boldsymbol{t}\ \&\ \mu_t \vDash \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{t'}))$

Let us introduce into the language two *modal-temporal* operators:

"historical necessity" $\Box_t$:
(8) $\mu_t \vDash \Box_t \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{t})$ iff $\forall \boldsymbol{\beta} \in \boldsymbol{W}_t(\boldsymbol{\alpha} \cong_t \boldsymbol{\beta} \Rightarrow \mu_t \vDash \boldsymbol{A}(\boldsymbol{\beta}, \boldsymbol{t}))$

and "historical possibility" $\Diamond_t$:
(9) $\mu_t \vDash \Diamond_t \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{t})$ iff $\exists \boldsymbol{\beta} \in \boldsymbol{W}_t(\boldsymbol{\alpha} \cong_t \boldsymbol{\beta}\ \&\ \mu_t \vDash \boldsymbol{A}(\boldsymbol{\beta}, \boldsymbol{t}))$

Further, define a special *deontic-temporal* operator $\boldsymbol{O}_t$:

(10) $\boldsymbol{O}_t \boldsymbol{A} = \Box_t \boldsymbol{A} \vee \Box_t \neg \boldsymbol{A}$.

This is an expression of some kind of historical *predetermination*. It tells that independently of human actions and efforts the state of affairs described by means of the proposition ***A*** either obtains or does not obtain in every world from some set of worlds which share one and the same history.

Further, note that if condition (10) fails to obtain this means that we are dealing with the formulation of genuine norms. Evidently, there are 27 conjunctions of the forms ***G'A***, ***F'A***, and $\neg$***F'A***. Only seven of from them are essentially different:

G'A which allows us to define a deontic modality "it is obligatory";
(***G'A*** $\vee$ ***F'A***) which expresses "it is favourable";
(***G'A*** $\vee$ $\neg$***F'A***) which expresses "it is either obligatory or forbidden";
F'A which expresses "it is permissible";
(***F'A*** $\vee$ $\neg$***F'A***) which expresses "it is either permissible or forbidden";
$\neg$***F'A*** which expresses "it is forbidden (not-permissible)";
(***G'A*** $\vee$ ***F'A*** $\vee$ $\neg$***F'A***) which expresses "it is indifferent in normative sense".

And there is also a form ***F''A***, i.e. "it is permissible at the appointed time".

Evidently, completeness and decidability of the tensed logical system $\boldsymbol{K}_b$ apply to a system of deontic logic with the above-mentioned modalities.

The part of the system which meets the requirement for the operators $\Box_t$ and $\Diamond_t$ is at least the system ***S5***. Its "mixed" schemes of provable formulae are in particular:

DT.1 $\Box_t \boldsymbol{A} \rightarrow \boldsymbol{OA}$; this scheme corresponds to the condition $\boldsymbol{R}_t$;
DT.2 $\boldsymbol{OA} \rightarrow \Diamond_t \boldsymbol{A}$; this scheme also corresponds to the condition $\boldsymbol{R}_t$ and represents a stronger version of the Kantian principle: "ought" implies "can".
DT.3 $\Box_t \boldsymbol{A} \leftrightarrow \boldsymbol{O} \Box_t \boldsymbol{A}$ and
DT.4 $\Diamond_t \boldsymbol{A} \leftrightarrow \boldsymbol{O} \Diamond_t \boldsymbol{A}$; according to them statements about obligations, if they concern "historically necessary" or "historically possible" states of affairs, turn out to be empty statements.

The operator defined by the condition (1) allows us to affirm that propositions about historically predetermined states of affairs are, in normative sense, empty ones.

DT.5 $\boldsymbol{O}_t \boldsymbol{A} \rightarrow (\boldsymbol{A} \leftrightarrow \boldsymbol{OA})$

Statement: Let $\boldsymbol{\Omega}$ is any (sensible) temporal modality of past time, i.e. something expressed in our formalized language by means of any finite and non-empty sequence of operators ***H*** and ***P***. Then if a formula ***A*** does not contain entries of operators $\Box_t$, $\Diamond_t$ and ***O*** then the proposition $\boldsymbol{O}_t \boldsymbol{\Omega A}$ is true.

On the grounds of the above-mentioned statement and the thesis DT.5 it is possible to conclude that judgements expressed by means of propositions of past time always will be, in normative sense, empty ones. When a formula ***A*** does not contain entries of operators $\Box_t$, $\Diamond_t$ and ***O*** the proposition $(\boldsymbol{\Omega A} \leftrightarrow \boldsymbol{O}_t \boldsymbol{\Omega A})$ is true. It is quite clear: obligations imposed and accepted at present and pertinent to past events do not have any normative meaning. Indeed, "You ought (today) to present at yesterday's meeting" does not mean much more than "May be, you presented at yesterday's meeting".

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