Mathematical Sense: Wittgenstein's Syntactical Structuralism

VICTOR RODYCH, LETHBRIDGE

1. Introduction

Stated dogmatically, on Wittgenstein's unorthodox view of mathematics, only algorithmically decidable concatenations of 'signs' are mathematical propositions and only *proved* mathematical propositions have mathematical *sense* ('Sinn'). If, e.g., we do not know of an applicable and effective decision procedure for deciding Goldbach's Conjecture (hereafter 'GC') in any existent mathematical calculus, GC is *not* a mathematical proposition, and if we prove GC in, say, Peano Arithmetic (PA) tomorrow, GC is a *new* mathematical proposition with a *new* sense in a *newly* created calculus, PA_2.

Stated equally dogmatically, on Wittgenstein's unorthodox view of mathematics, only 'axioms' and *proved* concatenations of 'signs' are mathematical propositions with mathematical *sense* (hereafter, "concatenation(s) of 'signs'" will be abbreviated as 'Csign(s)'). Not only is GC not a mathematical proposition before it is decided, "28 × 76 = 2228" is *also* not a mathematical proposition before it is decided by a decision procedure, and when it is 'refuted' it is still not a mathematical proposition.

These two incompatible articulations of Wittgenstein's position reflect, as we shall see, Wittgenstein's (apparently) incomplete ruminations on a number of inter-connected issues. Each account has drawbacks for Wittgenstein's core view, but, as we shall see, the only substantive difference is that, on the second articulation, both *undecided* algorithmically decidable Csigns and algorithmically refuted Csigns are *not* mathematical propositions.

It should go without saying that neither articulation of Wittgenstein's account is the received view of a mathematical proposition and its sense or meaning.

On the received view, a mathematical conjecture, such as GC, is a genuine mathematical proposition, with a fully determinate sense (meaning) and, possibly, a fully determinate truth-value, even if (a) GC is independent of all existent mathematical calculi and/or (b) we do not have in hand an applicable and effective decision procedure by which to decide it. GC's sense (or meaning) is, simply: Every even number greater than 2 is the sum of two primes. On the received view, GC says (or we use GC to assert) something *about every even number*. We understand GC insofar as we are able to understand this meaning – insofar as we are able to understand what it is, or what it would be like, for every even number greater than 2 to be the sum of two primes (i.e., GC's truth conditions). As a matter of fact, we *do* understand GC because we understand its truth conditions.[1]

This paper aims to (1) show *how* Wittgenstein's radical position on a mathematical proposition and its sense results from his life-long view that mathematics is exclusively syntactical and invented by human beings, (2) *propose* a particular conception of "mathematical sense" – and an interpretation of Wittgenstein's remarks – that best resolves the internal tension between two of Wittgenstein's principal views on mathematics (which are detailed in Section 3), and (3) consider some objections to Wittgenstein's view and how he does or might respond to them.

Wittgenstein's reasoning about mathematical sense is best understood in connection with the rest of his radical constructivist philosophy of mathematics. According to this view, human beings invent mathematics *bit-by-little-bit*, which means, in part, that we don't discover *pre-existing* proofs – they exist only when we have constructed them. Furthermore, on Wittgenstein's account: (a) mathematical propositions do not speak about – and are not used by us to make assertions about – infinitely many objects in a real or possible world[2]; (b) the set of natural numbers is not an infinite extension, but rather a recursive rule for enumerating the naturals; and (c) the so-called provable propositions of, say, PA do not exist already as a set of 'theorems' (i.e., it is not a *fact*, today, that GC is provable in PA, even if GC is proved in PA tomorrow). Though Wittgenstein argues for (a)-(c), we will only here see part of his argument for (a).[3]

2. Propositions, Sense, and Mathematics in the *Tractatus*

As is well known, one of the main aims, if not *the* main aim, of the *Tractatus* is to work out the language-reality connection by determining what is required for language, or language *usage*, to be *about* the world. Wittgenstein's two-pronged, core position in the *Tractatus* is that there is only one reality (i.e., "the world")[4] and propositions, which have sense ('Sinn') and which are used by us to make assertions about the world (e.g., to picture or represent a possible state of affairs or a possible fact),[5] are (or must be) either true or false because they must either agree with the world or disagree with the world (4.022, 4.25, 4.062, 2.222).

Though Wittgenstein abandons some Tractarian positions, he always maintains that the only genuine propositions that we can use to make assertions about reality are 'empirical' (contingent) propositions, which are true if they agree with reality and false otherwise.

Wittgenstein's core Tractarian position immediately yields a conception of mathematics, mathematical propositions, and "mathematical truth." Since there is only one type of genuine proposition and only one type of genuine truth, it follows that all other apparent propositions are pseudo-propositions of various types and that all other uses of 'true' and 'truth' deviate markedly from the truth-by-correspondence (or agreement) that contingent propositions have in relation to reality.

In the *Tractatus*, Wittgenstein clearly states (6.2) that "[t]he propositions of mathematics are equations, and therefore pseudo-propositions" – that "[a] proposition of mathematics does not express a thought" (6.21), has no sense, and therefore cannot be used by us to assert that a possible state of affairs or fact exists.[6] Moreover, "in order to tell whether a picture is true or false we must compare it with reality" (2.223),[7] from which it follows that "[i]t is impossible to tell from the picture alone whether it is true or false" (2.224), which means that "[t]here are no pictures that are true a priori" (2.225). Thus, if there is something we call "mathematical truth", it is not agreement with reality and it is not a kind of *a priori* truth (whatever that would be). As Wittgenstein says at (6.2321), "the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness."

3. Mathematical Propositions and Mathematical Sense after 1928

When he returns to Philosophy in 1929, Wittgenstein elaborates his Tractarian position by saying that "mathematical propositions so called are not propositions at all" (*LWL* 13) and that "there are no true a priori propositions" (*LWL* 13).[8] Indeed, from the *Tractatus* until the end of his life, Wittgenstein maintains that "mathematical propositions" are not real propositions and that "mathematical truth" is essentially non-referential and purely syntactical in nature.

In presenting a positive account of mathematical sense, the intermediate Wittgenstein endeavours to:

1. Distinguish mathematical questions from mathematical problems.
2. Demarcate mathematical propositions from Csigns that are *not* mathematical propositions.
3. Ascertain the relationship between mathematical sense and decidability.
4. Connect mathematical sense to decision procedures.
5. Ascertain the relationship between mathematical sense and mathematical proof.
6. Ascertain whether undecided and refuted "mathematical propositions" have mathematical sense.

To these ends, the middle Wittgenstein *ruminates* in various, evidently incompatible ways and these ruminations provide material for at least the following four positions on mathematical sense.

Strong Verificationism: The sense of a mathematical proposition *is* its proof (i.e., "the verification… is *the* sense of the proposition") (*PR* §166; *WVC* 227).[9]

Weak Verificationism$_1$: The sense of a mathematical proposition is (a) *determined by* its proof, which (b) gives the mathematical proposition a *new* sense (c) in a *new* calculus ((a) "[t]he verification [of a mathematical proposition]… *determines* the sense of the proposition" (*PG* 458-459; italics mine)[10]; "the proof belongs to the *sense* of the proved proposition, i.e. determines that sense" (*PG* 375); (b) "in proving it we give it a new sense that it didn't have before" (*PG* 374)[11]; (c) "a mathematical proof incorporates the mathematical proposition into a new calculus" (*PG* 371)).

Weak Verificationism$_2$: A Csign constitutes a mathematical proposition – which *has* mathematical sense – *if and only if* it is algorithmically decidable in an existent mathematical calculus *and* we know this to be the case (*PR*

§§149, 151; *PG* 366, 452). The sense of a mathematical proposition is *determined by* or *corresponds to* a decision procedure (i.e., "[t]he method of checking corresponds to the sense of the mathematical proposition" (*PG* 366; 458-459); "it isn't as if it were only certain that a mathematical proposition made sense ["has a sense"][12] when it (or its opposite) had been proved", for "[t]his would mean that its opposite would never have a sense (Weyl)" (*PR* §148)).

Structuralism: The sense of a mathematical proposition is its syntactical *location* within a calculus and its syntactical connections within that calculus (i.e., "a mathematical proposition is only the immediately visible surface of a whole body of proof and this surface is the boundary facing us" (*PR* §162); "the properties of a number are the properties of a position" (*PG* 457)).[13]

In attempting to distinguish mathematical questions from mathematical problems (#1) and demarcate mathematical propositions from Csigns that are *not* mathematical propositions (#2), Wittgenstein argues (##3, 4, 6) that algorithmically decidable Csigns are mathematical propositions with sense and that refuted propositions *have* sense,[14] since "[i]t obviously makes sense to say 'I know how you check ["36 x 47 = 128"]', even before you've done so" (*PR* §153). The merit of this position – Wittgenstein's Weak Verificationism$_2$ – is that it clearly defines a mathematical proposition as a Csign that is algorithmically decidable in an existent calculus. The problem, however, is that it does not say *what* the sense of a mathematical proposition *is*. In particular, Weak Verificationism$_2$ says that an undecided mathematical proposition, which is algorithmically decidable, *has sense* (i.e., since it is a *meaningful* or genuine mathematical proposition), whereas Weak Verificationism$_1$ precludes *undecided* (or perhaps *unproved*) mathematical propositions from having sense.

4. The Tension in Wittgenstein's Account and Three Incompatible Resolutions

In trying to understand Wittgenstein's account of mathematical sense, we need not be detained by Strong Verificationism, for Wittgenstein does not maintain this position for very long even in the middle period. The *tension*, however, between Weak Verificationism$_1$ and Weak Verificationism$_2$ exists throughout the middle period and there is evidence that it remains in the later period (e.g., *RFM* V, §9, 1942). To resolve this tension, we must find one or more reasonable interpretations that recognize that a proof gives a "mathematical proposition" "a new sense that it didn't have before" (*PG* 374), which requires that we drop the Weak Verificationism$_2$ claim, most notable at (*PR* §148), that algorithmically decidable Csigns have sense *before* they are decided. This *type* of resolution yields the following three incompatible interpretations of the main strands of Wittgenstein's account.

Weak Verificationist$_1$ Structuralism (WV$_1$S)

(A_1): Mathematical Proposition: A Csign is a mathematical proposition *of* calculus Γ *iff* it is algorithmically decidable *in* calculus Γ and we know this to be the case.

(B_1): Having Mathematical Sense: Only primitive propositions (e.g., axioms) and proved propositions of calculus Γ *have* mathematical sense in calculus Γ.[15]

(C_1): The Sense of a Mathematical Proposition of Calculus Γ: *is* its syntactical position in the syntactical structure that is calculus Γ.

According to WV$_1$S, the *sense* that a particular proposition φ of Γ has in Γ is its exact syntactical location in that syntactic structure, which, in part, consists of the syntactical connections φ has with other propositions of Γ, as mediated by the syntactical rules of Γ.

Weak Verificationist$_2$ Structuralism (WV$_2$S)

(A_2): Mathematical Proposition: A Csign is a mathematical proposition *of* calculus Γ *iff* it is algorithmically decidable *in* calculus Γ and we know this to be the case.

(B_2): Having Mathematical Sense: All and only propositions *decided* in calculus Γ have mathematical sense in calculus Γ.

(C_2): The Sense of a Mathematical Proposition of Calculus Γ: *is* its syntactical position in the syntactical structure that is calculus Γ (if proved in Γ) or its syntactical *conflict* with a proved proposition in calculus Γ (if refuted).

Weak Verificationist$_1$ Structuralism (WV_1S_2)[16]
(A_3): Mathematical Proposition: A Csign is a mathematical proposition of calculus Γ *iff* either (i) it is a "primitive proposition" (e.g., an axiom) of calculus Γ or (ii) it is a proved proposition in calculus Γ (*RFM* App. III, §6).
(B_3): Having Mathematical Sense: Only primitive propositions (e.g., axioms) and proved propositions of calculus Γ *have* mathematical sense in calculus Γ.
(C_3): The Sense of a Mathematical Proposition of Calculus Γ: *is* its syntactical position in the syntactical structure that is calculus Γ.

5. A Preference for WV_1S

Given his ##1-6 aims, the best and simplest way for Wittgenstein to resolve the aforementioned tension, I believe, is to adopt WV_1S.

WV_1S is preferable to WV_1S_2 in that WV_1S captures Wittgenstein's use of the Law of the Excluded Middle and algorithmic decidability (see, e.g., *PR* §151, cf. *PR* §§173-174; *PG* 400) as conjoined criteria that define a mathematical proposition relative to a calculus. In the middle period and, it seems, also in the later period, Wittgenstein repeatedly states that a Csign is a mathematical proposition in a particular calculus *iff* the Law of the Excluded Middle applies to it, which means nothing more than we know how to decide it by means of a decision procedure. The problem with WV_1S_2 is that, according to it, a Csign for which we have in hand an applicable and effective decision procedure is *not* a mathematical proposition before it is decided and, if it is refuted, it is still not a mathematical proposition. This conflicts with Wittgenstein's insistence that if we know how to decide a Csign because we recognize that our decision procedure applies to the Csign's syntactical structure, then that Csign is a mathematical proposition (i.e., by virtue of its syntactical structure).

As regards WV_2S, WV_1S is preferable to it for two reasons. First, WV_1S better accommodates Wittgenstein's Structuralism, for if the mathematical sense of a mathematical proposition *is a syntactical position in a structure* (i.e., a syntactical calculus), a Csign that does *not* have a position in a mathematical structure (i.e., a calculus), such as a refuted 'proposition,' cannot have mathematical sense. Second, WV_1S better accommodates Wittgenstein's Weak Verificationism$_1$ claims that a *proof* – *not* a proof *and* a refutation – "belongs to the *sense* of the proved proposition" (*PG* 375), "give[s] it a new sense that it didn't have before" (*PG* 374), and "incorporates the mathematical proposition into a new calculus" (*PG* 371). These three claims about a proof and the sense of the proved proposition go hand-in-hand with Wittgenstein's Structuralist claim that the sense of a mathematical proposition is its syntactical *location* within a calculus, for a proved proposition is incorporated *into* a new calculus (and the syntactical connections among its propositions), but a refuted 'proposition' is not part of the proof-syntactical structure of a mathematical calculus. Indeed, so-called 'ill-formed' Csigns (e.g., "2 + = 2 = 4") and syntactically *independent* Csigns are just as much *not* part of the proof-syntactical structure of a mathematical calculus as a refuted mathematical proposition (e.g., "2 + 2 = 5"). As Wittgenstein says (MS 163, 46v-47r, 1941):

> But I always want to say: true and false in mathematics corresponds, in the application to propositions of experience, not to the opposition true-false, but to the distinction between sense and nonsense.

Only a syntactically well-constructed contingent proposition has contingent sense; a nonsensical pseudo-proposition, such as "Socrates is identical" (*Tractatus*, 5.473, 5.4733), does not have sense *because* it is syntactically ill-constructed. A 'true' (e.g., proved) mathematical proposition corresponds to a contingent proposition with sense insofar as both are syntactically well-constructed: the mathematical proposition, *because of its syntactical form*, has a position in a syntactical calculus; the contingent proposition, because of *its* syntactical form, has sense and can be used to assert that a possible state of affairs or fact obtains. A 'false' (i.e., refuted) "mathematical proposition" corresponds to a nonsensical, contingent pseudo-proposition inasmuch as both are syntactically ill-constructed: the refuted mathematical proposition, *because of its syntactical form*, does not have a position in any existent syntactical calculus; the nonsensical pseudo-proposition, because of *its* syntactical form, does not have sense and cannot be used to make an assertion. Thus, just as contingent sense depends on "syntactical correctness", mathematical sense similarly depends upon "syntactical correctness": If a mathematical proposition has sense it is part of a mathematical calculus (i.e., as an 'axiom' or a proved proposition) because of its syntactical form – if a mathematical proposition does not have sense, it is not part of a mathematical calculus because of its syntactical form. In the mathematical case, bad or incorrect syntax means that a Csign – whether a refuted mathematical proposition, an ill-formed Csign (e.g., "2 + = 2 = 4"), or a syntactically independent Csign – is not a mathematical proposition with mathematical sense and a syntactical location in a mathematical calculus. What we call 'truth' and 'falsity' in mathematics is just as much a matter of syntax as sense and nonsense in the realm of contingent propositions. If we recognize this, it should not be disconcerting that even refuted mathematical *propositions*, such as 37 + 63 = 101", do not have mathematical sense, since they simply are not components, with syntactical connections to other propositions, of syntactical mathematical structures. Seen in this light, WV_2S's claim (C_2) that the sense of a refuted mathematical proposition is its syntactical *conflict* with a proved proposition in a mathematical calculus has no substance, and flies in the face of Wittgenstein's structuralism, since such a proposition has no syntactical connections with propositions *in* the calculus.

Although WV_1S has the non-standard consequence that undecided and refuted mathematical propositions do *not* have mathematical sense, it has the decided merit of agreeing with *Wittgenstein's* (intermediate) view that "one cannot discover any connection between parts of mathematics or logic that was already there without one knowing" (*PG* 481), which Wittgenstein clearly maintains in the later period, saying, e.g., that "the proof… makes new connexions", "[i]t does not establish that they are there", for "they do not exist until it makes them" (*RFM* III, §31). In proving a proposition in a mathematical calculus, one makes or constructs new syntactical connections; given that the sense of a mathematical proposition consists in these very connections, a mathematical proposition cannot have sense until we have constructed these connections.

> A proof is a proof of a particular proposition if it goes by a rule correlating the proposition to the proof. That is, the proposition must belong to a system of propositions, and the proof to a system of proofs. And *every proposition in mathematics must belong to a calculus of mathematics*. (*PG* 376; italics mine)

This, then, is Wittgenstein's account of mathematical sense. It is, I believe, the most coherent interpretation of Wittgenstein's numerous and lengthy remarks on mathematical invention, construction, proof, decidability and, most importantly, sense. The question, of course, is whether this radical conception is plausible and defensible.

Superficially, at least, the answer seems to be a quick ‘No,’ for WV_1S seems to have the revisionist consequence that Csigns that we ordinarily take to be (meaningful) mathematical propositions, such as GC, are *not* mathematical propositions, given that they are not algorithmically decidable. This seems especially problematic and revisionistic, for if GC is not a mathematical proposition, what good (mathematical) reason could we have to even attempt to decide GC relative to any mathematical calculus?

The apparent revisionism, however, is more a matter of how we talk than a matter of Wittgenstein prohibiting certain mathematical activities. To see this, suppose, e.g., that a mathematician tackles GC because s/he wants to determine whether or not it can be proved using only the axioms and rules of PA. Does Wittgenstein’s conception prohibit such an attempt? Wittgenstein explicitly says that it does *not*.

> I do not claim that it is wrong or illegitimate if anyone concerns himself with Fermat's Last Theorem. Not at all! If e.g. I have a method for looking integers [sic] that satisfy the equation $x^2 + y^2 = z^2$, then the formula $x^n + y^n = z^n$ may stimulate me. I may let a formula stimulate me. Thus I shall say, Here there is a *stimulus* – but not a *question*. Mathematical ‘problems’ are always such stimuli. (*WVC* 144)[17]

The fact that, according to WV_1S, GC is not a mathematical proposition of PA does not, in itself, prohibit us from using PA to decide GC. If a mathematician succeeds in proving GC using only the axioms and rules of PA, the received view says that s/he has proved GC *in* PA, whereas Wittgenstein claims that s/he has created a new, extended calculus, PA_2, in which GC (or, more accurately, a proved inductive base and a proved inductive step) is a proved proposition (propositions) with sense. On both views, a mathematician successfully proves a Csign using only the axioms and rules of PA – the only difference, and it is important, is that on Wittgenstein’s view, the Csign (GC) was not a proposition of PA before or after the proof.

More generally, on Wittgenstein’s account (WV_1S), the mathematician can decide algorithmically decidable mathematical propositions and s/he can still endeavour to decide Csigns that are not algorithmically decidable. In the latter case, s/he may or may not be successful, and if s/he is successful, s/he may extend a mathematical calculus by adding a newly proved Csign to a calculus, thereby creating a new mathematical calculus. In the former case, s/he will determine whether a mathematical proposition is part of an extended calculus and, typically, even if s/he refutes a mathematical proposition, s/he will add a newly proved mathematical proposition to a newly created (i.e., extended) mathematical calculus. In both cases, the mathematician can decide Csigns and, in some cases, extend an existent calculus (i.e., *create* a new, extended calculus) in exactly the ways that mathematicians in fact decide 'propositions' and extend calculi. What seems like revisionism is, therefore, really only a difference in terminology,[18] for a mathematician has no less reason to try to prove GC using only the axioms and rules of PA.

The real, substantive difference between WV_1S and the received view of mathematics is that, according to WV_1S, we do not *pretend* that any and all 'well-formed' Csigns (i.e., so-called 'wffs') are mathematical propositions with determinate senses and truth-values and we don't pretend that we can stipulate well-formedness for the 'propositions' of undecidable mathematical calculi. For example, GC is a well-formed formula of PA, but, even on the standard account, we simply do not know whether or not it will be proved independent of PA. If GC is one day proved independent of PA, this will only clarify the delusion that we are able to stipulate rules for well-formedness for mathematical calculi that lack an applicable and effective decision procedure. Gödel's First Incompleteness Theorem should already have taught us this – for Gödel has constructed a procedure by which we can generate *wffs* of PA which are syntactically independent of PA if PA is consistent – but, instead, the significance of Gödel's First Incompleteness Theorem supposedly lies in a demonstration that mathematical truth and proof are distinct.

What seduces us into thinking that GC *is* a mathematical proposition with a fully determinate sense is a "faulty analogy" between mathematical and contingent propositions. According to this seductive analogy, just as contingent/empirical propositions are about *existent* objects or phenomena, all mathematical propositions are *about* existent (or possible) mathematical objects, and so, just as the former have determinate sense because they assert that some or all objects of kind A have property B, mathematical propositions similarly assert that some or all mathematical objects of kind A have property B. On this, the received view, a mathematical proposition such as GC, which says that all objects of kind A (i.e., all even numbers greater than 2) have property B (i.e., are the sums of two primes), has a determinate sense simply because it ascribes a property to existent (or possible) objects. Even though GC is undecided and we do not know how to decide it, GC has a completely determinate sense which we *completely* understand, because "it says" that every even number greater than 2 *is* the sum of two primes and we know what the words 'every,' 'even' and 'prime' mean in mathematics.[19] We believe that we can envisage infinitely many even numbers each of which is the sum of two primes; *that*, we say, *is* GC's sense (or meaning), whether or not it *turns out* to be true or false (indeed, whether or not we ever decide it).

According to Wittgenstein, however, this picture of mathematical sense is entirely mistaken because it rests on a misleading analogy between (genuine) propositions and so-called mathematical propositions (*PG* 370-71), which engenders the "false picture" (*PG* 290) that mathematical propositions have a 'sense' and/or are '*about*' something. Just as he says in the *Tractatus* that "[o]ne can understand [a proposition]… without knowing whether it is true" (4.024), Wittgenstein says at (*PG* 370-71) that "the discovery that a particular [truth determination] hypothesis [i.e., "the quantity of haemoglobin in the blood… diminishes according to such and such a law in proportion to the time after death"] is true (or: agrees with the facts)" 'does not change anything in the grammar of the proposition "the man died two hours ago".' One can understand "the man died two hours ago" *completely* without knowing whether it is true because, given our linguistic conventions, this sentence has a fully determinate sense which we can understand, picture, etc. Moreover, we can *describe* the aforementioned "possible method" of "ascertaining the time of death", we can "ascertain[] experimentally… whether the description corresponds to the facts", and, if it does, we can thereby 'medically' *prove* that the man died two hours ago, without "incorporat[ing] the hypothesis… proved into any new calculus" and so without "giv[ing] it any new sense" (*PG* 371). However, unlike the description of the hypothesized time-of-death-determination method, "the mathematical proof couldn't be described before it is discovered", which shows that we are mistaken to "take the discovery of a proof in mathematics, sight unseen, as being the same or similar" to such a "medical proof" (*PG* 371). In the case of an undecided mathematical proposition, such as GC, the 'proposition' does not have a sense because we cannot picture or describe what is the case if it is true (i.e., we cannot picture or describe GC's truth-conditions) and we cannot picture or describe what would be the case if it were proved (i.e., we cannot picture or describe GC's proof-conditions in the absence of a proof). We erroneously think we fully *understand* GC because we erroneously think that this type of 'proposition' is on a par with a proposition about reality. According to Wittgenstein, we can escape the trap of this misleading analogy only if we recognize that *understanding* a mathematical proposition is not possible independent of a calculus and a proof, for mathematical understanding is inextricably linked to a proposition's *having* mathematical sense, which requires that it is *already* proved.

6. Mathematical Sense and Mathematical Understanding

From 1929 through 1944, Wittgenstein makes this point – that mathematical proof, sense, and understanding are inextricably connected – in myriad different (and interesting) ways. Contra the naïve view of GC whereby we can fully understand its sense in its English (i.e., natural language) expression, Wittgenstein says (*PR* §162) that "[a] mathematical proposition – unlike a genuine proposition – is *essentially* the last link in a demonstration that renders it visibly right or wrong."[20] We cannot understand GC until it has a sense, and it does not have a sense until we have given it a sense by proving it in a calculus. Since nothing exists in mathematics unless and until it is *constructed*, a proof does not discover a pre-existing mathematical fact, which means that "when I learn the proof ["that there are infinitely many primes"], I learn something *completely new*, and not just the way leading to a goal with which I'm already familiar" (*PR* §155).[21] Only when we have constructed a proof does the proof of a proposition exist, and only then does the proposition have a mathematical sense because only then does it have a connected syntactical place in a syntactical structure.

> Only within ["an extended technique of calculating with cardinal numbers"] does this proposition ["there are infinitely prime numbers"] *have sense* [italics mine]. A proof of the proposition locates it in the whole system of calculations.
>
> …
>
> The proof of a proposition certainly does not mention, certainly does not describe, the whole system of calculation that stands behind the proposition *and gives it its sense*" (italics mine). (*RFM* VI, §11)[22]

In what remains of this paper, I will briefly present one of the many ways in which the middle and later Wittgenstein connects *understanding* a mathematical proposition and the sense of a mathematical proposition and I will consider how Wittgenstein does and would respond to standard objections to his conception of mathematical sense.

In 1929, when Wittgenstein writes (*PR* §155; MS 108, 13) that "when I learn the proof ["that there are infinitely many primes"], I learn something *completely new*", he also writes (MS 105, 57-59) that "the real mathematical proposition is a proof of a so-called mathematical proposition" and that "[t]he real mathematical proposition is the proof: that is to say, the thing which shows how matters stand" (*PR*, p. 184, Ft. #1). Similarly, 12 years later, at (*RFM* VII, §§10, 11), Wittgenstein asks: "[O]ught I to say… that when a proof is found the sense alters?" Wittgenstein immediately rejoins, on behalf of the received view, that such a view is absurd, for it follows that '"the proof of a proposition cannot ever be found, for, if it has been found, it is no longer the proof of *this* proposition".'[23] "But", replies Wittgenstein, "to say this is so far to say nothing at all", for what exactly is *this* proposition? One only finds his viewpoint strange, Wittgenstein thinks, because one assumes that one has everything one can possibly have in terms of a proposition when one has the undecided proposition, but what we actually see is that a proof extends a calculus and makes new connections that did not previously exist (and with which we can *only now* work). "[T]he proof belongs to the *sense* of the proved proposition, i.e. determines that sense", Wittgenstein argues (*PG* 375), "[i]t isn't something that brings it about that we believe a particular proposition, but something that shows us *what* we believe – if we can talk of believing here at all."

Wittgenstein repeatedly contrasts mathematical sense and mathematical understanding with our ability to understand a contingent proposition at first sight and without knowing its truth-value. In the *Tractatus*, Wittgenstein asserts that "[a] proposition is a picture of reality", "for if I understand a proposition, I know the situation that it represents" – "[t]o understand a proposition means to know what is the case if it is true" (4.024). One can understand a proposition that one has never seen or heard by knowing its object names and their respective *Bedeutung* and knowing the conventional linguistic rules for 'composing' propositions out of names. Ultimately, though, one *understands by representing* in one's mind – by, e.g., '*thinking*' (*Tractatus* 3.11) – a possible state of affairs or fact. Wittgenstein *contrasts* this conception of understanding a contingent proposition with understanding a so-called "mathematical proposition", which, he argues, we *cannot* understand or picture in advance of its proof (or decision).

In 1944, the later Wittgenstein continues this investigation.

> Would one say that someone understood the proposition '563 + 437 = 1000' if he did not know how it can be proved? Can one deny that it is a sign of understanding a proposition, if a man knows how it could be proved?
>
> The problem of finding a mathematical decision of a theorem might with some justice be called *the problem of giving mathematical sense to a formula*. (*RFM* V, §42; italics mine)

Just before this passage in MS 127, Wittgenstein writes (MS 127, 161; March 4, 1944): "If the Fermat proposition were proved to me, I would then understand it better afterwards than before." A little later, Wittgenstein rhetorically asks (MS 127, 171-172): "Don't I understand the Fundamental Theorem of Algebra better if I can prove it than if I cannot prove it? How can it be that the proof does not contribute to my understanding, since it surely shows for the first time where this proposition is at home?"

At (*RFM* VI, §13), Wittgenstein again attacks his position with the received view of understanding the sense of an undecided mathematical proposition. His interlocutor asks: "Now isn't it absurd to say that one doesn't understand the sense of Fermat's last theorem?" – "Don't [the mathematicians] *understand* it just as completely as one can possibly understand it?" "But", Wittgenstein rejoins, "if I am to know what a proposition like Fermat's last theorem says, must I not know what the criterion is, for the proposition to be true?" "I am of course acquainted with criteria for the truth of *similar* propositions", he adds, "but not with any criterion of truth of this proposition." Wittgenstein's point is that, although we are familiar with criteria of truth/*proof* for similar propositions, unless and until Fermat's Last Theorem is proved, we do not know (and we cannot describe) *its unique* truth/*proof*-conditions (i.e., what proof proves it 'true'). When Fermat's Last Theorem is *proved*, something new is created – syntactical connections are established with other parts of the calculus – and only then do we fully understand the sense of Fermat's Last Theorem, because only then does Fermat's Last Theorem *have* a determinate sense. Indeed, only when a proposition is proved *is it* a 'machine-part' with 'connexions' to other proved machine-parts (i.e., propositions) of the machine (calculus) (*RFM* VI, §13).

The categorical difference between truth-conditions for contingent propositions and so-called truth (or proof) conditions for mathematical propositions is fundamental here. We cannot picture or imagine (or believe) the *mathematical sense* of a mathematical conjecture such as GC unless and until it is proved because it is not a genuine, *referential* proposition – we cannot picture GC's sense (or meaning) because GC, like all mathematical propositions, is not about a realm of entities and, therefore, its truth-conditions are, more precisely, *proof-conditions*. The sense of a contingent proposition can be understood without knowing its truth-value, simply because one can know precisely what would make it true (i.e., its truth-conditions) without knowing *if* it is true; the sense of a mathematical proposition *cannot* be understood without knowing its "truth-value", simply because one *cannot* know *precisely* what would make it true (proved) without knowing whether *and how* it is true (proved).

Wittgenstein makes the same point in terms of the *use* of a mathematical proposition. In connection with the "issue whether an existence-proof which is not a construction is a real proof of existence" (*RFM* V, §46), 'the question arises: Do I *understand* the proposition "There is…" when I have no possibility of finding where it exists?' Wittgenstein answers that "in so far as what I can do with the proposition is the criterion of understanding it, thus far it is not clear *in advance* whether and to what extent I understand it." Here Wittgenstein alludes to his life-long methodological principle, first articulated at *Tractatus* 6.211: "In philosophy the question, 'What do we actually use this word or this proposition for?' repeatedly leads to valuable insights." If you want to understand an expression or proposition, or if you want to know what an expression or proposition *means*, ask *how* (or determine how) it is *used*. Not only is it "a sign of understanding a [mathematical] proposition… if [one] knows how it could be proved" and a sign of no understanding "if [one does] not know how it can be proved", if *no one* knows how to use a mathematical proposition (or whether it can be used), that is a decisive sign that it does not have a determinate sense. "Let the use of words teach you their meaning ['Bedeutung']", Wittgenstein says as late as 1949 (MS 144, pp. 91-94; *PI,* 2001 Edition, p. 187e); "[s]imilarly one can often say in mathematics: let the *proof* teach you *what* was being proved."

7. Infinitistic Mathematical Conjectures

Though Wittgenstein wishes to destroy the misleading analogy between (genuine) propositions and so-called mathematical propositions (*PG* 370-71), he does grant that the analogy is strongest when the mathematical proposition in question is finitistically restricted, as, e.g., if we assert $GC_{1,000,000}$ (i.e., "All even numbers greater than 2 and less than or equal to 1,000,000 are the sums of two primes"), for in this case we know how to algorithmically decide $GC_{1,000,000}$. Even in the finite case, however, the analogy nevertheless breaks down, Wittgenstein argues, because in deciding $GC_{1,000,000}$ we may give it a sense that it didn't have before it was decided (i.e., if we prove it). Where, however, the analogy is truly dangerous and misleading is in the infinite case, where we might contrast, "There are infinitely many chairs" with "There are infinitely many prime numbers." As Wittgenstein says:

> If you tried to say 'There are infinitely many chairs' in the way in which you can say 'There are infinitely many prime numbers,' then your statement would not be false; it would be senseless. For there is no way of verifying this statement. (*WVC* 227-228)

We simply cannot establish the truth of "There are infinitely many chairs", though we can empirically establish the truth of "There are *n* chairs" for an arbitrarily large *n*. Much to the contrary, we *can* establish the 'truth' of "There are infinitely many prime numbers" (by mathematical induction, in Wittgenstein's restricted, constructive sense[24]), but as with physical objects such as chairs, we cannot do this empirically. Thus, on Wittgenstein's view, if we abandon the explanatorily useless idea of a world (or domain) of, e.g., infinitely many natural numbers, we are *forced* to grant that the so-called truth of, say, GC *resides in a proof*, not in infinitely many even numbers *having* a particular property. The sense or meaning of GC, therefore, is not comparable to an empirical proposition that ascribes a property to one or more existent objects – it only has sense if and when it is proved (constructed) by a constructed proof.

8. Conclusion

Put still differently, Wittgenstein's main criticism of the received view on mathematical sense is that *when* the sense of an infinitistic mathematical conjecture, such as GC, is understood, we do *not* come to know the individual senses of infinitely many conjuncts, such as "8 is the sum of 5 and 3" and "10 is the sum of 7 and 3" and "12 is the sum of 7 and 5", etc., as we *can* when we decide $GC_{1,000,000}$. If we come to *know* that GC is 'true,' we won't come to *know*, for each even number (of infinitely many), which two primes sum to it – we will come to *know*, as with Euclid's Prime Number Theorem, *why* every even number greater than 2 *must* be the sum of two primes, for we will see, *by way of a proof*, the syntactical connections that ensure this so-called 'necessity,' because they *constitute* a particular structure.[25] Only with a proof can we understand *why and in what exact sense* a proposition is 'true' because we see, syntactically, why the syntactical sense of the proposition is so – i.e., we see in a proof of GC *why* or *how* each *constructible* even number greater than two is or must be the sum of two primes – we see *how* we can construct, without limit or exception, even numbers that are the sums of two primes.

In sum, the sense of a mathematical proposition does not exist until it is knowingly constructed by a proof, which is why the unnecessary concept of "mathematical truth" is, unlike its contingent counterpart, *not* distinct from meaning, understanding, and construction. We simply cannot understand a mathematical proposition, which *does not* have a determinate sense/meaning, unless and until it is proved. This deviant conclusion is not so bizarre if we, following Wittgenstein, *reject mathematical possibility as actuality or a type of reality.*[26]

References

Ambrose, Alice: 1982, "Wittgenstein on Mathematical Proof", *Mind*, 91, No. 362: 264-372.

Clark, Peter, and Hale, Bob (Eds.): 1994, *Reading Putnam*, Cambridge, Mass.: Blackwell Publishers.

Coliva, A. & Picardi, E. (Eds.): 2004, *Wittgenstein Today*, Padova, Il Poligrafo.

Dummett, Michael: 1973, "The Philosophical Basis of Intuitionistic Logic", reprinted in Dummett (1978), pp. 215-247.

Dummett, Michael: 1978, *Truth and Other Enigmas*, Cambridge, Mass.: Harvard University Press.

Dummett, Michael: 1994, "Wittgenstein on Necessity: Some Reflections", in Clark and Hale (1994), pp. 49-65.

Dummett, Michael: 1997: Review of Pasquale Frascolla's *Wittgenstein's Philosophy of Mathematics* (New York: Routledge, 1994), and Christoffer Gefwert's *Wittgenstein on Philosophy and Mathematics* (Abo: Abo Akademi University Press, 1994), in *Journal of Philosophy*, p. 359-374.

Floyd, Juliet: 2005, "Wittgenstein on Philosophy of Logic and Mathematics", in Shapiro (2005), pp. 75-128.

Frascolla, Pasquale: 1994, *Wittgenstein's Philosophy of Mathematics*, London and New York: Routledge.

Frascolla, Pasquale: 2004, "Wittgenstein on Mathematical Proof", in A. Coliva & E. Picardi (2004), pp. 167-184.

Lee, Desmond, (Ed.): 1980, *Wittgenstein's Lectures, Cambridge 1930-32: From the Notes of John King and Desmond Lee*, Oxford: Basil Blackwell; referred to in the text as '*LWL*.'

Marion, Mathieu: 1998, *Wittgenstein, Finitism, and the Foundations of Mathematics*, Oxford: Clarendon Press.

Marion, Mathieu: 2004, "Wittgenstein on Mathematics: Constructivism or Constructivity?", in Coliva & Picardi (2004), pp. 201-222.

Marion, Mathieu: 2006, "Wittgenstein on Mathematical Proof", unpublished paper delivered at the University of Kent, January 2006.

Potter, Michael: 2006, "Wittgenstein's Later Philosophy of Mathematics: Is There Any Such Thing?", unpublished paper delivered at the University of Kent, January 2006.

Rodych, Victor: 1997, “Wittgenstein on Mathematical Meaningfulness, Decidability, and Application”, *Notre Dame Journal of Formal Logic*, Vol. 38, Number 2, pp. 195-224.

Rodych, Victor: 1999a, “Wittgenstein’s Inversion of Gödel’s Theorem”, *Erkenntnis*, Vol. 51, Nos. 2/3, pp. 173-206.

Rodych, Victor: 1999b, “Wittgenstein on Irrationals and Algorithmic Decidability”, *Synthese*, Vol. 118, No. 2, pp. 279-304.

Rodych, Victor: 2000a, “Wittgenstein’s Critique of Set Theory”, *The Southern Journal of Philosophy*, Vol. 38, No. 2, pp. 281-319.

Rodych, Victor: 2000b [published January 2003], “Wittgenstein’s Anti-Modal Finitism”, *Logique et Analyse*, Vol. 43, No. 171-172, pp. 301-333.

Rodych, Victor: 2002, “Wittgenstein on Gödel: The Newly Published Remarks”, *Erkenntnis*, Vol. 56, No. 3, pp. 379-397.

Rodych, Victor: 2003, “Misunderstanding Gödel: New Arguments about Wittgenstein and New Remarks by Wittgenstein”, *Dialectica*, Vol. 57, No. 3, pp. 279-313.

Rodych, Victor: 2006, “Who Is Wittgenstein’s Worst Enemy?: Steiner on Wittgenstein on Gödel”, *Logique et Analyse*, Vol. 49, No. 193: 55-84.

Rodych, Victor: 2007, “Wittgenstein’s Philosophy of Mathematics”, *Stanford Encyclopedia of Philosophy*, February 23, 2007; <http://plato.stanford.edu/archives/spr2007/entries/wittgenstein-mathematics/>.

Shapiro, Stewart (ed.): 2005, *The Oxford Handbook of Philosophy of Logic and Mathematics*, Oxford: Oxford University Press:

Waismann, Friedrich: 1979, *Wittgenstein and the Vienna Circle*, edited and translated by B. F. McGuinness (Oxford: Basil Blackwell); referred to in the text as ‘*WVC*.’

Wittgenstein, Ludwig: 1922, *Tractatus Logico-Philosophicus*, London: Routledge and Kegan Paul, 1961; translated by D.F. Pears and B.F. McGuinness.

Wittgenstein, Ludwig: 1975, *Philosophical Remarks*, Oxford: Basil Blackwell; Rush Rhees (Ed.); translated by Raymond Hargreaves and Roger White; referred to in the text as ‘*PR*.’

Wittgenstein, Ludwig: 1974, *Philosophical Grammar*, Oxford: Basil Blackwell; Rush Rhees (Ed.); translated by Anthony Kenny; referred to in the text as ‘*PG*.’

Wittgenstein, Ludwig: 1978 [1956], *Remarks on the Foundations of Mathematics*, Second Edition, Oxford: Basil Blackwell; edited by G. H. von

Wright, R. Rhees and G. E. M. Anscombe; referred to in the text as '*RFM*.'
Wittgenstein, Ludwig: 2000, *Wittgenstein's Nachlass*: *The Bergen Electronic Edition*, Oxford: Oxford University Press.

Notes

1. This reasoning *must* have it that we similarly understand Fermat's Last Theorem, even though there are very few people in the world that understand Andrew Wiles' proof. In his (1973) account of Intuitionistic mathematics, "which leans heavily upon Wittgensteinian ideas about language" (226) and in which "use exhaustively determines meaning" (pp. 218, 220, 223), Michael Dummett says (p. 225) that "a grasp of the meaning of a [mathematical] statement consists in a capacity to recognise a proof of it when one is presented to us." As we will see in Sections 5-8, however, Wittgenstein's position is perhaps stricter than Dummett's account. An examination of the similarities and differences between Wittgenstein's position and Dummett's, however, is beyond the scope of the present paper.
2. Indeed, for Wittgenstein (*PG* 290), "mathematics is a calculus and hence isn't really *about* anything." Cf. (*PR* §§109, 157, 159), (*WVC* 106), (*PG* 333, 468), and (*PG* 468): "In mathematics *everything* is algorithm and *nothing* is meaning ['Bedeutung']."
3. For the details of my interpretation of Wittgenstein's Philosophy of Mathematics, see (Rodych 1997, 1999a, 1999b, 2000a, 2000b, 2002, 2003, 2006, 2007). For different approaches to the problem of Wittgenstein on mathematical sense, see (Ambrose 1982) and (Frascolla 2004). I would like to thank Mathieu Marion and Michael Potter for kindly sending me copies of (Marion 2006) and (Potter 2006), respectively.
4. On the ontological side, Wittgenstein says, most fundamentally, that "[t]he world is all that is the case" (1) and that the world is "[t]he totality of existing states of affairs" (2.04). Given that "[a] state of affairs… is a combination of ['simple,' 'unalterable' (2.02; 2.023; 2.2027] objects (things)" (2.01) and given that "a fact-is the existence of [two or more] states of affairs" (2), it follows that "[t]he world is the totality of facts" (1.1).
5. On the propositional side, a proposition "is a picture of reality" (4.01; 4.021; 4.032), "[w]hat a picture represents is its sense" (2.221), and "[a] proposition can be true or false only in virtue of being a picture of reality" (4.06). According to Wittgenstein, both representational drawings and linguistic representations are pictures in that both have a requisite *structural* aspect *and* a requisite *intensional* aspect: to represent a possible state of affairs or a possible fact, a picture must be *intended* to represent that possible state of affairs or possible fact *and* it must be isomorphic with that possible state of affairs or possible fact. As Wittgenstein says, "[w]e picture [possible] facts to ourselves" (2.1) – " 'A state of affairs is thinkable'… means… that we can picture it to ourselves" (3.001) – and "[w]e use the perceptible sign of a proposition (spoken or written, etc.) as a projection of a possible situation", wherein "[t]he method of projection is to *think* of the sense of the proposition" (3.11; italics mine). Thus, the *sense* of a proposition is a *possible* state of affairs or fact (4.031), which one can picture (or think or project) mentally/cognitively. Though the later Wittgenstein's was reluctant to discuss the ontological status of thinking

(and mental states and events in general), if this picturing or thinking is a brain event, this is perfectly compatible with the monism-physicalism of the *Tractatus*.

6. Mathematical pseudo-propositions *do not have sense*, which we can picture, think, or project, and hence, they cannot be true or false by agreeing or failing to agree with an existent state of affairs or fact in the world. See (*AWL* 197) on *imagining* a mathematical proposition and (*LFM* 123) and (*RFM* I, §§106-112) on *believing* a mathematical proposition.
7. See also (4.05): "Reality is compared with propositions."
8. See also (*LWL* 1): "Language consists of propositions (excluding for the moment so-called mathematical propositions). A proposition is a picture of reality, and we compare proposition [sic] with reality." Cf. (*LWL* 2).
9. See also (*PR* §154): "What a mathematical proposition says is always what its proof proves"; "it never says more than its proof proves." Cf. (*PR* §163): "If we want to see what has been proved, we ought to look at nothing but the proof"; and (*PG* 369).
10. In the (*PG* 458-459) passage (from MS 113, May 23, 1932), Wittgenstein rewords the (*PR* §166) passage (MS 107, Sept. 11, 1929) and more weakly states that "[t]he verification [of a mathematical proposition]… *determines* the sense of the proposition."
11. Cf. (*PR* §153): "Understanding *p* means understanding its system. If *p* appears to go over from one system into another, then *p* has, in reality, changed its sense." (*PG* 378) is almost identical.
12. A better translation of this passage would use "has a sense" rather than "made sense." See Note #14, below. (*PR* §148) is perhaps the most illuminatingly ruminative passage in *PR*, revealing the tensions in Wittgenstein's evolving thoughts. Cf.: "If there is no method provided for deciding whether the proposition is true or false, then it is pointless, and that means senseless" (*PG* 451); "[W]hat *would* mean nothing would be to say that I can only assert ["a mathematical proposition"] if it's correct" (*PR* §150). See also (*PR* §202). Indeed, in this connection, the middle Wittgenstein speaks of "mathematical truth" and "mathematical falsehood": "For, in a very important sense, every significant proposition must teach us through its sense how we are to convince ourselves whether it is true or false" ((*PR* §148); cf. (*PG* 366)), whereas the later Wittgenstein (like the early Wittgenstein of the *Tractatus*) usually speaks only of 'correct', 'incorrect' and 'proved' mathematical propositions.
13. The later Wittgenstein similarly says that the sense of a mathematical proposition is a 'position' (*RFM* VI, §11) or its 'place' (*RFM* VII, §10) as a "machine-part" with "connexions" in a machine-calculus (*RFM* VI, §13)). See also (*RFM* III, §§27, 29).
14. It should be noted that translations in *PR* and *PG* frequently have "made sense" or "make sense" when literal and better translations would be "had sense" and "have sense", which also agree with Wittgenstein's use of (mathematical) 'sense' as a technical term. See, e.g., (*PR* §148, par. 3; §150, par. 15; §153, par. 3). I would like to thank Dr. Tim Pope (University of Lethbridge) for the translations contained herein. Any errors in the translations are entirely my responsibility.
15. Cf. (Marion 1998, 173): "For Wittgenstein, proof not only is the truth-maker, it is, so to speak, the meaning-maker."

16. It seems that Wang (1991, 253, 256) finally arrived at this interpretation of Wittgenstein's Philosophy of Mathematics, as referenced in my (1999a). See also (Dummett 1994, 50, 56, 63-64) and, reviewing (Frascolla 1994), (Dummett 1997, 363, 366, 369).
17. Cf. (*PG* 371): "(Unproved mathematical propositions [Csigns] – signposts for mathematical investigation, stimuli to mathematical constructions.)"
18. See (Rodych 2000b), especially pp. 258-267, for Wittgenstein's views on mathematical induction, unsystematic proof searches, and extensions of calculi.
19. Cf. (*PG* 375).
20. Wittgenstein says (*PR* §155) that "it isn't the prose which is the mathematical proposition, it's the exact expression", the sequence of signs and its syntactical relations to other propositions in its calculus. (*PG* 369-370): "If you want to know what the expression "continuity of a function" means, look at the proof of continuity; that will show what it proves."
21. "[W]e don't discover a proposition like the fundamental theorem of algebra,… we merely construct it…, [b]ecause in proving it we give it a new sense that it didn't have before" (*PG* 374). Cf. (*RFM* VI, §11).
22. The first sentence has been re-translated by Dr. Tim Pope, with the principle difference being that "makes sense" is here replaced by "have sense." Cf. (*RFM* III, §25): "A psychological disadvantage of proofs that construct *propositions* is that they easily make us forget that the *sense* of the result is not to be read off from this by itself, but from the *proof*. In this respect the intrusion of the Russellian symbolism into the proofs has done a great deal of harm."
23. Cf. (*PR* §155): "But in that case it's unintelligible that I should admit, when I've got the proof, that it's a proof of precisely *this* proposition, or of the induction meant by this proposition."
24. See (Rodych 2000b).
25. Just as we came to see in (one variant of) Euclid's Theorem, *not* the infinite distribution of the primes, but that there cannot be a greatest prime number, since for any prime n, there must be a greater prime in the interval $[n + 2, n! + 1]$.
26. I would like to express my appreciation for supportive funding from the Social Sciences and Humanities Research Council of Canada.