DEFINITE DESCRIPTIONS: LANGUAGE, LOGIC, AND ELIMINATION

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Abstract
Definite descriptions are in the focus of philosophical discussion at least since Russell's famous paper "On Denoting". In this paper, we present a logic with descriptions in Russell's spirit. The formulation, however, is closely related to Schütte's development of predicate logic, i.e. the formulation of the calculus uses positive- and negative-parts. With respect to this slightly more sophisticated formulation it is possible to formalize Russell's convention which is originally stated in the meta-language of his theory of descriptions within our calculus. In this paper we prove an elimination theorem for this calculus.

1 INTRODUCTION

Russell's theory of definite descriptions is an intellectual offspring of finding (adequate) solutions to puzzling philosophical oddities, e.g. for Meinong's ontological jungle. In this article we focus on a calculus with descriptions that is very much in the spirit of Bertrand Russell's treatment of descriptions in the *Principia Mathematica* *14 (*P.M.**14). However, it is not the same. This is due to several reasons:

(a) Russell's theory of descriptions is not focussed on the whole class of descriptions but just on descriptions of a very simple kind.
(b) Russell does not develop his theory in a purely formal way. Essential parts of it are found in the context that explains some important facts, e.g. the use of the scope-operator.
(c) The language and logic (developed below) is void of intensional functors.

According to Russell a description has the following form:

the so-and-so

where 'the' is in the singular and 'so-and-so' is a (possibly) complex expression. (Definite) descriptions occur in two contexts – as Russell says, e.g.

- The present King of France is wise.
- The present King of France exists.

Following Russell, the above statements can be formally represented as follows:

(*14.01) $[\iota x A(x)]B(\iota x A(x)) \leftrightarrow \exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x))$
(*14.02) $E!\iota x A(x) \leftrightarrow \exists x \forall y(A(y) \leftrightarrow x = y)$

Kurt Gödel remarks (1944, p.126) in his contribution to the Schilpp-volume that honours Russell:

> It is to be regretted that this first comprehensive and thorough going representation of a mathematical logic and the derivation of Mathematics from it is so greatly lacking in formal precision in the foundations (contained in *1–*21 of *Principia*), that it represents in this respect a considerable step backward as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs, in particular in connection with "incomplete symbols".[1]

Russell licenses the inference from (*14.01) to

(*14.101) $B(\iota x A(x)) \leftrightarrow \exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x))$

with the following convention:

> It will be found that, in most cases in which descriptions occur, their scope is, in practice, the smallest proposition enclosed in dots or brackets in which they are contained. […] For this reason it is convenient that, when the scope of an occurrence of $(\iota x)(\phi x)$ is the smallest proposition enclosed in dots or other

[1] Interestingly, B. Russell did not respond to Gödel in the Schilpp-volume.

brackets, in which the occurrence in question is contained, the scope need not be indicated by "$[(\iota x)(\phi x)]$." […] This convention enables us in the vast majority of cases that actually occur, to dispense with the explicit indication of the scope of a descriptive symbol; […]

In the light of the quite critical remarks of Gödel, we develop in this paper a *thorough going* treatment of descriptions.

At several occasions Russell points out that definite descriptions have *no meaning in isolation* and that definite descriptions are *incomplete symbols*. In our view the phrase "no meaning in isolation" is a semantical thesis and is treated elsewhere (e.g. Gratzl (submitted)). In this paper I shall focus on the phrase "incomplete symbol", that means – according to our understanding – that there is to each formula A containing a definite description a formula B without any definite descriptions and the formulas A and B are provable equivalent relative to a certain translation. The details of this formal interpretation of "incomplete symbols" will be stated below – in section 4.

2 LANGUAGE

The alphabet of L consists of a denumerable set of free individual variables, a denumerable set of bound individual variables, a denumerable set of n-ary predicates variables, the logical symbols: $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$, ι, […], =, E! and the auxiliary signs (and). We use as syntactical variables a, b, c for free individual variables; x, y, z for bound variables; u, v for any term and P^n, Q^n, R^n, S^n for any n-ary predicate variable.

2.1 Definition (Formulas and Terms of L)

1. Free individual variables are terms.
2. If $a_1,\ldots, a_n$ are terms and P^n is an n-ary predicate variable, then $P^n(a_1,\ldots, a_n)$ is a formula.
3. If a, b are terms, then $(a = b)$ is a formula.
4. If A is a formula, then $\neg A$ is a formula.
5. If A, B are formulas, then $(A \wedge B)$, $(A \vee B)$ are formulas.
6. If $A(a)$ is a formula, such that the bound individual variable x does not occur in it, then $\forall x A(x)$ and $\exists x A(x)$ are formulas.
7. If $A(a)$ and $B(b)$ (a and b are not necessarily distinct) are formulas such that the bound individual variable x does not occur

in them, then $\iota xA(x)$ is a term and $[\iota xA(x)]B(\iota xA(x))$ and $B(\iota xA(x))$ are formulas.

8. If $\iota xA(x)$ is a term, then $E!\iota xA(x)$ is a formula.
9. If $[\iota xA(x)]B(\iota xA(x))$ and $[\iota xC(x)]D(\iota xC(x))$ are formulas, then: $[\iota xA(x), \iota xC(x)](B(\iota xA(x)) \wedge D(\iota xC(x)))$ and $[\iota xA(x), \iota xC(x)](B(\iota xA(x)) \vee D(\iota xC(x)))$ are formulas.[2]
10. If $\iota xA(x, a)$ is a term and $B(a)$ is a formula, then $\forall y([\iota xA(x, y]B(\iota xA(x, y)))$ and $\exists y([\iota xA(x, y]B(\iota xA(x, y)))$ are formulas.
11. Nothing else is a formula or a term.

The remaining connectives are defined as usual.

2.2 A brief note on E!

Russell writes in the *PM* on p.174 (Russell/Whitehead 1970):

> When in ordinary language or in philosophy, something is said to "exist," it is always something *described*, *i.e.* it is not something immediately presented, like a taste or a patch of colour, but something like "matter" or "mind" or "Homer (meaning "the author of the Homeric poems"), which is known by description as "the so-and-so," and is thus of the form $(\iota x)(\phi x)$. Thus in all such cases, the existence of the (grammatical) subject $(\iota x)(\phi x)$ can be analytically inferred from any true proposition having this grammatical subject. It would seem that the word "existence" cannot be significantly applied to subjects immediately given; *i.e.* not only does our definition give no meaning to "E!x," but there is no reason, in philosophy, to suppose that a meaning of existence could be found which would be applicable to immediately given subjects.

Without entering the epistemological distinction between *knowledge by acquaintance* and *knowledge by description* we take it from this quote for granted, that for Russell the E!-predicate can only sensibly be applied to ι-terms. This is reflected in the definition of terms and formulas – condition 8.

[2] This clause simply allows multiple occurrences of ι–terms in the scope. Although we shall not deal explicitly with such formulas in this article.

2.3 Examples

The following two examples should indicate that even quite simple examples taken from natural language show that the logical structure might already have a certain complexity, e.g. ι-terms can *overlap* in such a way that the bound variable of a ι-term occurs in the basis of another one. The first example is:

- The first born child of its father inherits the farm.

I use the following symbols:

$P(x, y)$ stands for: y is father of x
$Q(x, y)$ stands for: x is a firstborn child of z
$R(x)$ stands for: x inherits the farm

So the the sentence ist formalized as:

$$R(\iota x Q(x, \iota y P(x, y)))$$

The second example is:

•• The first born child of its father inherits the fatherly farm.

This quite easily understandable sentence will be expressed in the language developed above, by using the following symbols:

$P(x, y)$ stands for: y is father of x
$Q(x, z)$ stands for x is a firstborn child of z
$R(x, z)$ stands for: x inherits z
$S(z, y)$ stands for: z is farm of y

'The first born child of its father inherits the fatherly farm.' then is formalized as:

$$R(\iota x Q(x, \iota y P(x, y)), \iota z S(z, \iota y P(z, y)))$$

2.4 Definition (Positive and negative parts of formulas)

In section 3 we shall state an extended form of a Schütte-style calculus for predicate logic including rules dealing with definite descriptions. Schütte's formulation of the predicate calculus makes use of positive and negative parts; those should be thought of generalized notions of antecendent (i.e. negative parts) and and consequent (i.e. positive parts) of Gentzen's sequents.

The following definition of positive and negative parts of formulas is due to Schütte (1960, p.11).

1. F is a positive part of F.
2. If $\neg A$ is a positive part of F, then A is a negative part of F.
3. If $\neg A$ is a negative part of F, then A is a positive part of F.
4. If $(A \vee B)$ is a positive part of F, then both A and B are positive parts of F.
5. If $(A \wedge B)$ is a negative part of F, then both A and B are negative parts of F.

In the following presentation of (ιC) we make use of subscript plus und subscript minus: these are devices to mark positve- and negative-parts within formulas.

3 ι-Calculus (ιC)

Axioms

(ι1) $F[A_+, A_-]$
(ι2) $F[(a = a)_+]$
(ι3) $F[(a = b)_-, A(a)_-, A(b)_+]$

Rules of inference

(ιR1) $F[A_+], F[B_+] \Rightarrow F[(A \wedge B)_+]$
(ιR2) $F[A_-], F[B_-] \Rightarrow F[(A \vee B)_-]$
(ιR3) $F[A(a)_+] \Rightarrow F[(\forall x A(x)_+]$

(ιR4) $F[A(a)_{-}] \Rightarrow F[(\exists x A(x)_{-}]$[3]
(ιR5) $F[(\forall x A(x))_{-}] \vee \neg A(a) \Rightarrow F[(\forall x A(x))_{-}]$
(ιR6) $F[(\exists x A(x))_{+}] \vee A(a) \Rightarrow F[(\exists x A(x))_{+}]$
(ιR7) $F[(a = b)_{+}], G[(a = b)_{-}] \Rightarrow F[\ldots] \vee G[\ldots]$
(ιR8) $F[([\iota x A(x)]B(\iota x A(x)))_{+}] \Rightarrow F[(B(\iota x A(x)))_{+}]$
(ιR9) $F[(B(\iota x A(x)))_{+}] \Rightarrow F[(\exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x)))_{+}]$
(ιR10) $F[(B(\iota x A(x)))_{-}] \Rightarrow F[([\iota x A(x)]B(\iota x A(x)))_{-}]$
(ιR11) $F[([\iota x A(x)]B(\iota x A(x)))_{-}] \Rightarrow F[(\exists x \forall y((A(y) \leftrightarrow x = y) \wedge B(x)))_{-}]$
(ιR12) $F[(E!\iota x A(x))_{+}] \Rightarrow F[(\exists x \forall y(A(y) \leftrightarrow x = y))_{+}]$
(ιR13) $F[(E!\iota x A(x))_{-}] \Rightarrow F[(\exists x \forall y(A(y) \leftrightarrow x = y))_{-}]$

(ι1)–(ι3) and (ιR1)–(ιR8) are the usual axioms and rules of inference of predicate logic with equality. The rules (ιR8)–(ιR13) deal with definite descriptions and are formulated in such a way that Russell's "contextual definitions" are provable.

Provability is defined as follows: (i) Every axiom is provable, and (ii) if the premises of a rule of inference are provable, then the conclusion of this rule of inference is provable. It is easily seen that Russell's so-called *contextual definitions* are provable in (ι*C*),. i.e. (*14.01, *14.02, and *14.101).

4 Elimination of ι-terms

4.1 Definition (ι-rank, rank)

The number of occurrences of the ι-symbol in a given formula is called the ι-rank (ι*rk*) of this formula. The number of logical signs in a formula is the rank (*rk*) of this formula.

4.2 Inductive definition of F relative to F*

(i) If $\iota rk(F) = 0$, then F^* is F.
(ii) If $\iota rk(F) > 0$, then:
  (a) If F is $B(\iota x A(x))$ where $\iota x A(x)$ is the leftmost ι-term, and B is not of the form $\neg C$, $[\iota x A(x)]C$, $[\iota x A(x)]\neg C$, then F^* is $(\exists x \forall y((A^*(y) \leftrightarrow x = y) \wedge B^*(x)))$.

[3] Both (ιR3) and (ιR4) are subject to the conditon on variables, i.e. the free individual variable *a* must not occur in the conclusion.

(b) If F is $E!\iota xA(x)$, then F^* is $\exists x\forall y(A(y) \leftrightarrow x = y)$.

(c) If F is $\neg B(\iota xA(x))$, where $\iota xA(x))$ is the leftmost ι-term, then F^* is $\exists x\forall y((A^*(y) \leftrightarrow x = y) \wedge \neg B^*(x))$.

(d) If F is $\neg[\iota xA(x)]B(\iota xA(x))$, where $\iota xA(x)$ is the leftmost ι-term, then F^* is $\neg(\exists x\forall y((A^*(y) \leftrightarrow x = y) \wedge B^*(x)))$.

(e) If F is $[\iota xA(x)]\neg B(\iota xA(x))$, where $\iota xA(x)$ is the leftmost ι-term, then F^* is $\exists x\forall y((A^*(y) \leftrightarrow x = y) \wedge \neg B^*(x))$

(f) If F is $[\iota xA(x)]B(\iota xA(x))$, where $\iota xA(x)$ is the leftmost ι-term, then F^* is $\exists x\forall y((A^*(y) \leftrightarrow x = y) \wedge B^*(x))$

4.3 Elimination Theorem

If $(\iota C) \vdash F$ with $\iota rk(F) \geq 0$, then there is a formula F^* relative (to the inductive definition stated above), such that $(\iota C) \vdash F \leftrightarrow F^*$.

In effect this theorem expresses that everything that can be said with the aid of ι-terms can be stated in predicate logic with equality (lets call it (C) without any loss, i.e. (ιC) is a conversartive extension of (C). However, the *reduced* formula, i.e. the formula obtained by elimination of ι-terms, may be quite cumbersome to read.

Proof sketch: The proof is an induction on the number of ι-terms, i.e. ι-rank, of a formula F. If $\iota rk(F) = 0$, then there is nothing to prove. If $\iota rk(F) > 0$, then several cases (according to the translation *) have to be considered. This proof-step – although every step is quite easy to prove – will be illustrated with an example:

$(\iota C) \vdash [\iota xA(x)]B(\iota xA(x) \leftrightarrow \exists x\forall y((A(y) \leftrightarrow x = y) \wedge B(x))$ such that $\iota xA(x)$ is the left-most ι-term.

1. $[\iota xA(x)]B(\iota xA(x) \rightarrow [\iota xA(x)]B(\iota xA(x)$ (ι1)
2. $[\iota xA(x)]B(\iota xA(x) \rightarrow B(\iota xA(x)$ (1., (ιR8))
3. $[\iota xA(x)]B(\iota xA(x) \rightarrow \exists x\forall y((A\ (y) \leftrightarrow x = y) \wedge B(x))$ (2., (ιR9))
4. $[\iota xA(x)]B(\iota xA(x) \rightarrow [\iota xA(x)]B(\iota xA(x)$ (ι1)
5. $\exists x\forall y((A(y) \leftrightarrow x = y) \wedge B(x)) \rightarrow [\iota xA(x)]B(\iota xA(x)$ (4., (ιR11))
6. $[\iota xA(x)]B(\iota xA(x) \leftrightarrow \exists x\forall y((A\ (y) \leftrightarrow x = y) \wedge B(x))$ (3., 5., Def. ↔)

By eliminating each ι-term from the left to the right in a given formula F the procedure eventually terminates.

5 CONCLUDING REMARKS

The elimination theorem for the ι-Calculus (ιC) is intended to state a formal interpretation of the phrase that definite descriptions are "incomplete symbols" – as Russell put it. Despite the constructive features of (ιC) it still needs proving – as Kripke (2005, p.1033) notes:

"In these cases, however, I recall proving – though it really takes proving! – that there are no real hydras. Every path eventually terminates, and all are equivalent."

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