

A PROOF OF NOMINALISM: AN EXERCISE IN SUCCESSFUL REDUCTION IN LOGIC

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Symbolic logic is a marvelous thing. It allows for an explicit expression of existence, viz. by means of the existential quantifier, and by it only. This is the true gist in Quine's slogan "to be is to be a value of a bound variable." Accordingly, one can also formulate explicitly the thesis of nominalism in terms of such logic. What this thesis says is that all the values of existential quantifiers we need in our language are particular objects, not higher-order objects such as properties, relations, functions and sets.

This requirement is satisfied by the first-order languages using the received first-order logic. The commonly used basic logic is therefore nominalistic. But this result does not tell anything, for the received first-order logic is far too weak to capture all we need in mathematics or science. According to conventional wisdom, we need for this purpose either higher-order logic or set theory. Now both of them deal with higher-order entities and hence violate the canons of nominalism. This does not refute nominalism, however. For I will show that both set theory and higher-order logic can be made dispensable by developing a more powerful first-order logic that can do the same job as they do.

Moreover, there are very serious problems connected with both of them. This constitutes an additional reason for dispensing with them in the foundations of mathematics. I will show how we can do just that. But we obviously need a better first-order logic for the purpose. Hence my first task is to develop one.

But is this a viable construal of the problem of nominalism? The very distinction between particular and higher-order entities might perhaps seem to be hard to capture in logical terms – harder than has been indicated so far. Logicians like Jouko Väänänen (2001) have emphasized the complexities involved in trying to distinguish first-order logic from higher-order logic. The first step of an answer is that the distinction cannot be made

purely formally but has to depend on the interpretation of one's logic, in particular on the specification of the values of bound variables.

But how does such an interpretation come about? This question should be generalized. How does a mathematical or scientific theory formulated in a logically explicit language work? The answer is obvious: By specifying a class of models. These models are the structures that the theory studies. In those models, the values of first-order quantifiers are particular objects (individuals) while the values of higher-order quantifiers are higher-order entities, such as sets, functions, predicates, relations etc.

But now the usual axiomatizations of set theory seem to belie the distinction. Such axiomatizations use the nominalistic first-order logic even though they are supposed to deal with sets, which are higher-order entities, albeit seemingly more concrete than properties and relations. The answer, which will be expounded more fully elsewhere, is that this is precisely what is wrong with first-order axiomatizations of set theory. (See here Hintikka, forthcoming (a) and (b).) They simply represent a wrong approach to set theory. They are based on a serious misunderstanding as to how an axiomatic theory works. Their models are structures of particulars, not structures of sets. Hence it is extremely difficult to try to extract information about structures of sets from a first-order axiomatization of set theory. Indeed, it can be explicitly proved that there cannot exist a set theory using the received first-order logic whose variables range over all sets.

Moreover, it is an important strategic defect of the usual axiomatic set theories that their logic is the usual first-order logic. For because of their reliance on conventional first-order logic it is impossible to define the concept of truth for such a set theory by means of its own resources. As a consequence one cannot discuss the model theory of set theory in set theory itself. This is a tremendous limitation foreshadowed in the so-called semantical paradoxes of set theory of yore. In this light, it is only to be expected that important questions concerning set-theoretical structures cannot be answered in the usual set theories. For instance the results by Cohen and Gödel concerning the unsolvability of the continuum problem on the basis of Zermelo-Fraenkel set theory thus only serve to confirm the reality of the predicted limitations.

Another mixed case seems to be obtainable by considering the higher-order logic known as type theory as a many-sorted first-order theory, each different type serving as one of the "sorts". One can try to interpret the logics of Frege and of Russell and Whitehead in this way. The attempt fails (systematically if not historically) because there are higher-

order logical principles that cannot be captured in terms of the usual formulations of many-sorted first-order logic. An important case is the so-called axiom of choice. (But see below how the axiom of choice can become a truth of a reformulated first-order logic). This example from set theory in fact reveals the crucial watershed between first-order logic and higher-order logic. It is not the style of variables, which only means pretending that one is dealing with this or that kind of entity, that is, first-order (particular) objects or higher-order ones. The crucial question is whether principles of inference are involved that go (or do not go) beyond the logical principles of first-order (nominalistic) logic.

These are vital problems to any serious thinker who tries to understand all mathematical reasoning nominalistically. A case in point is Hilbert. (See Hilbert (1918) and (1922).) Indeed, it is his nominalism that is largely responsible for Hilbert's having been labelled a "formalist". He wanted to interpret all mathematical thinking as dealing with structures of particular concrete objects. Now for mathematicians' deductions of theorems from axioms the interpretation of nonlogical primitives does not matter. In other words it does not matter what these objects are as long as they are particulars forming the right kind of structure. In this sense Hilbert could say that for the logical structure of his axiomatization of geometry he could have named his primitives "table", "chair" and "beer mug" instead of "line", "point" and "circle". One could not carry out such a reformulation of axioms and deductions from them if the values of a geometer's variables were entities which already have a structure like e.g. sets. A deduction is invariant with respect to a permutation of individuals but not of structures of individuals. One cannot hope to exchange the terms "triangle" and "quadrangle" in a geometrical proof and expect it to remain valid. Because of this invariance, Hilbert could say that the concrete particular objects in one of the models of his theory could be thought of as symbols and formulas. To use his own vivid language, Hilbert could have said that he could have named his geometrical primitives "letter", "word" and "formula", quite with the same justification as the envisaged terms "chair", "table" and "beer mug". This gambit was in fact put to use later as a technical resource by logicians, among them Henkin (1949) and Hintikka (1955) in building up the models they used to prove the completeness of the received first-order logic. It is thus a radical misunderstanding to label Hilbert a formalist

Hilbert blamed all the problems in the foundations of mathematics on the use of higher-order concepts. And he tried to practice nominalism and

not only profess it. He tried to show the dispensability of higher-order assumptions in mathematics. His test case was the controversial axiom of choice. Hilbert (1922) expressed the belief that in a proper perspective this “axiom” could turn out to be as obvious as $2+2=4$. Hilbert tried to accomplish this reduction to first-order level by replacing quantifiers by certain choice terms, so-called epsilon terms. These epsilon-terms are expressions of certain choice functions. Hilbert’s mistake was not to spell out what the choices in question depend on and thereby in effect ruling out some relevant kinds of dependence.

Hilbert did not succeed even though he was on the right track. There is in fact a far simpler way of showing that the axiom of choice can be understood as a first-order logical principle. All we need for the purpose is a slightly more flexible formulation of the usual first-order logic. One of its usual rules is existential instantiation that allow us to replace a variable x bound to a sentence-initial existential quantifier in a sentence $(\exists x)F[x]$ by a new individual constant, say β , at the same time as we omit the quantifier itself. This β can be thought of as a sample (some writers say “arbitrarily chosen”) representative of the truth making values of x . Hence such a β is like the legalese pseudo-names “John Doe” and “Jane Roe” representing existing but unknown individuals.

This rule cannot be applied to an existential quantifier $(\exists y)$ inside a formula, because the choice (sic) depends on the values of the universal quantifiers $(\forall z_1), (\forall z_2), \dots$ within the scope $(\exists y)$ occurs (assuming that the formula is in the negation normal form). But the rule of existential instantiation becomes applicable when we allow the substituting term to be a function term $\beta(z_1, z_2, \dots)$ which takes into account the dependencies in question. However, what is now introduced is not an individual constant, but a new function constant.

This reformulation of first-order logic is totally natural, and can be seen as following directly from certain eminently obvious truth-conditions of first-order (quantificational) sentences. The most obvious truth-condition for a first-order sentence S is the existence of suitable “witness individuals” that together show the truth of S . Thus for instance $(\exists x)F[x]$ is true iff there is an individual a that satisfies $F[a]$ and $(\forall x)(\exists y)G[x,y]$ is true iff for any given a there exists an individual b that satisfies $G[a,b]$. As this example shows, witness individuals may depend on other individuals. Hence their existence amounts to the existence of the functions (including constant functions) that yield as their values suitable witness individuals.

These functions are known as the Skolem functions of S . The functions β mentioned earlier are merely examples of “John Doe” Skolem functions.

This rule is a first-order one, for no higher-order quantifiers are involved. It seems to effect merely an eminently natural and eminently obvious reformulation of the rules of first-order logic. But when this reformulated first-order logic is used as the basis of second-order logic or set theory, the axiom of choice becomes a truth of logic without any further assumptions.

This result may at first seem too elementary to be of much interest. In reality, it puts the very idea of axiomatic set theory into jeopardy. In a historical perspective, Zermelo axiomatized set theory in the first place in order to defend his use of the axiom of choice in his proof of the well-ordering theorem. (See Zermelo 1908 (a) and (b), Ebbinghaus 2007.) We can now see that his axiomatizing efforts were redundant. Zermelo could have vindicated the axiom of choice by showing that it is a purely logical principle. First-order axiomatic set theory was right from the outset but logicians’ labor lost.

I am intrigued by the question why this exceedingly simply way of vindicating the status of the axiom of choice as a logical principle has not been used before. I suspect that the answer is even more intriguing. The new rule of functional instantiation is context sensitive, and hence makes first-order logic noncompositional. Now compositionality seems to have been an unspoken and sometimes even overt article of faith among logicians. It was what prevented Tarski from formulating a truth definition for a first-order language in the same language, as is shown in Hintikka and Sandu (1999). It might also be at the bottom of Zermelo’s unfortunate construal of the axiom of choice as a non-logical, mathematical assumption.

Systematically speaking, and even more importantly, the version of the axiom of choice that results from our reformulation is extremely strong. It is so strong that it is inconsistent with all the usual first-order axiom system of set theory. For instance, in a von Neumann-Bernays type set theory (Bernays 1968) it applies also to all classes and not only to sets. Accordingly, first-order set theories turn out to be inconsistent with (suitably formulated) first-order logic. This already shows that something is rotten in the state of first-order set theories.

But the axiom of choice is only the tip of the iceberg of problems (and opportunities) here. Earlier, I promised to develop a better first-order logic in order to defend nominalism. It turns out that we have to develop one in any case. The most fundamental insight here is that the received

first-order logic (logic of quantifiers) does not completely fulfill its job description. The semantical job of quantifiers is not exhausted by their expressing the nonemptiness and the exceptionlessness of (usually complex) predicates. But in the reconstructed axiom of choice another aspect of the meaning of quantifiers comes to play. By their formal dependence on each other, quantifiers also express the real-life dependence of the variables on each other that are bound to them. Such dependence is precisely what Skolem functions codify.

The formal dependence of a quantifier (Q_2y) on another quantifier (Q_1x) is in the received logic expressed by the fact that (Q_2y) occurs in the syntactical scope of (Q_1x). But this scope (nesting) relation is of a special sort, being among other features transitive and antisymmetric. Hence the received first-order logic cannot express all patterns of dependence and independence between variables. Since such dependence relations are the bread and butter of all science, this is a severe limitation of the received logic.

This defect is corrected in the independence-friendly (IF) logic that I have developed together with associates. (For it, see Hintikka 1996.) Its semantics is completely classical and can be obtained from the usual game-theoretical semantics simply by allowing our semantical games to be games with incomplete information.

The resulting logic is deductively weaker than the received first-order logic but richer in important ways in its expressive capacities. For instance, the equicardinality of two sets can be expressed by its means. Incidentally, this would have made it possible for Frege to define number on the first-order level, thus depriving him of one reason to use higher-order logic (as he does).

Because IF logic merely corrects a defect in the received “classical” first-order logic and because its semantics does not involve any new ideas (except independence, of course), it is not an alternative to the received first-order logic. It is not a special “nonclassical” logic or “alternative” logic. It is an improved version of the basic first-order logic. It replaces our usual logic.

In IF logic, the law of excluded middle does not hold, in spite of the classical character of its semantics. In other words, the negation \sim in it is not the contradictory negation but a strong (dual) negation.

An interesting feature of IF first-order logic is that its implications differ from those of the received first-order logic also when it comes to finite models. Since we have to replace the latter by the former, we must

also reconsider all finitary metamathematics and its prospects, including Hilbert's. This means that Hilbert's project has to be re-evaluated. It is no longer clear that Gödel's second incompleteness implies the impossibility of Hilbert's program. Indeed, there already exists an elementary proof of the consistency of an IF logic based elementary arithmetic in the same arithmetic. (See Hintikka and Karakadilar 2006.)

I will later show that IF first-order logic is as strong as the Σ_1^1 fragment of second-order logic (to be defined below). In simpler terms, second-order existential quantifiers are expressible in IF first-order logic.

This has major implications for the theme of this meeting which includes reduction. Reductions between theories are often implemented by mappings of the models of the reduced theory into the reduct theory. The existence of such a mapping is an existential second-order statement. If the theories in question are formulated by means of IF first-order logic, such reductions can be discussed in the same terms as theories themselves, which is impossible to do if the conventional first-order logic is used instead. What this means is that IF logic typically enables us to turn what used to be thought of as a metatheoretical examination of reductions into (object-language) scientific questions. Such algebras were studied already in Jónsson and Tarski (1951) and (1952).

IF logic can be extended unproblematically by introducing a sentence-initial contradictory negation \neg . The result can be called extended IF logic or EIF logic. This logic is the true new basic logic. Algebraically it is a Boolean algebra with an additional operator $\neg\sim$.

The contradictory negation $\neg S$ of S says that such winning strategies do not exist. Since semantical games are not all determinate, S may fail to be true and yet not necessarily be false. Thus $\neg S$ means game theoretically that for any strategy of the verifier there exists a strategy for the falsifier that defeats it if known to the falsifier. This can be expressed by a Π_2^1 second-order sentence.

One of the main advantages of IF logic is that it makes it possible to define truth for a sufficiently rich IF first-order theory in the same theory, thus overcoming the main handicap of conventional set theories. In terms of game-theoretical semantics, the truth-condition for a sentence S will say that there exists a winning strategy for the verifier in the semantical "verification" game $G(S)$ connected with S . This is a second-order condition, but it will be shown to reduce back to IF first-order logic. Technically, such winning strategies are codified in the Skolem functions explained earlier. Hence S is true if and only if its Skolem functions all exist. This can

be expressed by a second-order sentence with a string of second-order existential quantifiers followed by a first-order formula, i.e. by what is known as a Σ_1^1 sentence. They turn out to have IF first-order equivalents, as will be seen later in this paper.

Our new basic logic, the EIF first-order logic, is in many ways a highly interesting structure. It does not rely on the law of excluded middle and hence can be considered a realization of intuitionists' intuitions. Indeed, its systematic motivation makes it a more natural "intuitionistic logic" than the formal systems so-called. In spite of this essential kinship, it allows for much of the usual mathematics, including analysis. For instance, the unrestricted axiom of choice is valid in it. Precisely how much analysis can be handled by its means, is an important largely open problem to be investigated. This question means asking how much of analysis can be done by unproblematic elementary means.

EIF logic can be extended further by allowing \neg to occur also within the scope of quantifiers. The result can be called a fully extended IF logic (FEIF logic). This does not mean leaving the first-order level but requires further explanations (to be supplied later) for its semantics. It incorporates both the received first-order logic and IF logic as subsystems.

A clarifying comment is in order here. What precisely is the relation of the received first-order (RFO) logic to IF first-order logic, EIFFO logic and FEIFFO logic? The answer is that as long as *tertium non datur* holds, RFO logic and IFFO logic are analogous. RFO logic is the logic of those formulas for which the law of excluded middle holds. It is in this sense a part of IFFO logic. It is also that part of EIF logic whose formulas do not contain independence indicators (slashes) and for whose atomic sentences obey the law of excluded middle.

This FEIF logic can now serve the purpose of the reduction foreshadowed in my title. FEIF logic can be shown to be as strong as the entire second-order logic. And even though I have been speaking of higher-order logic, the fact is that second-order logic is strong enough to capture all the modes of reasoning used in mathematics and in science. In this sense, second-order logic is the true *Begriffsschrift* in Frege's ambitious terminology.

Second-order logic formulas are divided into fragments according to the number of changes of sign in their prenex form. Thus the $\Sigma_0^0 = \Pi_0^0$ fragment consists of first-order formulas, Σ_1^1 of formulas with a string of existential second-order quantifiers followed by a Σ_0^0 formula, and the Π_1^1 fragment consists of contradictory negations of Σ_1^1 formulas. In general, the Σ_{n+1}^1 fragment consists of a string of existential quantifiers of the order

$n+1$ followed by a formula in the Π_n^1 fragment. The Π_n^1 fragment consists of the contradictory negations of Σ_n^1 formulas.

The second-order logic that I propose to reduce to the first-order level is to be understood as the received “classical” second order logic. (It is this logic that is generally recognized as a sufficient medium of all normal mathematical reasoning.) That means that no independence indicator slashes occur in them. Consequently, the law of excluded middle holds. All sentences can be assumed in the negation normal form.

The reduction can be introduced by means of an example. Consider a Σ_1^1 sentence of the simple form $(\exists f)G[f]$ where $G[f]$ is a first order formula in its prenex form. Since $G[f]$ can be assumed to be an IF first-order formula, its prenex form can be assumed to begin with a string of universal quantifiers. In the simplest case, there is only one of them, say $(\forall x)$.

Let us assume also that f does not appear nested in (1). That means that it occurs in F only in contexts like $A(f(x))$ or $f(x) = a$. Hence (1) is of the form

$$(1) (\exists f)(\forall x)F[x, f(x)]$$

Now (1) is easily seen to be equivalent with the following formula

$$(2) (\forall x_1)(\forall x_2)(\exists y_1/\forall x_2)(\exists y_2/\forall x_1) \& ((x_1 = x_2) \supset (y_1 = y_2)) \& F^*$$

Here F^* is like F except that $A(f(x))$ has been replaced by $(\forall w)(w = y_1 \supset A(w))$ and $f(x) = a$ by $(\forall w)(w \supset y_1 \supset w = a)$.

This equivalence can be shown to hold as follows: The Skolem function translation into second-order form is

$$(3) (\exists f_1)(\exists f_2)(\forall x_1)(\forall x_2)((x_1 = x_2) \supset (f_1(x_1) = f_2(x_2))) \& F^{**}$$

where F^{**} is like F^* except that $f_2(x_2)$ replaces y_2 . Now the first conjunct in (3) says that f_1 and f_2 are the same function. Hence (3) and (2) are equivalent to (1).

This elimination is obviously repeatable so as to be extendable to any number of initial universal quantifiers in the prenex form of F and to any number of initial existential quantifiers $(\exists f_1)(\exists f_2), \dots$ instead of the single one $(\exists f)$.

If there is nesting of the functions f_1, f_2, \dots in F , it can be eliminated by introducing new functions f_i and new universal quantifiers $(\forall x_i)$. For

instance, if in F there occurs a term $f_1(f_2(x))$, we can add a new initial existential quantifier $(\exists g)$ and add after the prenex of F the conjunct

$$(4) (\forall z)(\forall u)((g(z) = u) \leftrightarrow (\exists w)(w = f_2(u) \ \& \ z = f_1(w)))$$

This procedure can be generalized. We can in fact easily replace predicate variables in the initial formula by variables for their characteristic functions. This extends the procedure so as to eliminate any formula-initial second-order existential quantifier albeit at the cost of introducing independence indicators (slashes). Moreover, this elimination can be applied to any unnegated existential quantifier in context, not just initial ones.

As to universal second-order quantifiers and negated second-order existential quantifiers, it suffices to point out that a universal second-order quantifier $(\forall f)$ prefixing a subformula $(\forall f)F[f]$ can be replaced by $\neg(\exists f)\neg F(f)$. We can now eliminate $(\exists f)$ from the subformula $(\exists f)\neg F[f]$ in the way just indicated. The result does not contain second-order quantifiers (if we apply the reduction from inside out) but does normally contain contradictory negations \neg . It is therefore a FEIF logic (sub) formula.

When these eliminations are carried out step by step in a given second-order formula, moving from inside out, all the second-order quantifiers are ultimately eliminated. The procedure does introduce independence indicators and contradictory negations, which is to say that the result is a FEIF formula, but does not introduce any second-order quantifiers. This accomplishes the reduction of the entire traditional second-order logic to the first-order level. The reduct sentence is a first-order sentence: all its quantifiers range over first-order entities. However, it does contain contradictory negations also within the scope of quantifiers. The additional power of second-order logic is therefore due, not to its being second-order, but to its unlimited use of contradictory negation. This means essentially unlimited use of tertium non datur. In a sense, this vindicates Brouwer's idea that the excessive power of classical mathematics stems from the use of the principle of excluded middle.

The same reduction can apparently be used to translate logic of the order $n+1$ to the order n and hence ultimately to the first-order level. This possibility will not be discussed here, however.

This procedure reduces the traditional second-order to the logic of the first-order level. However, what remains to be explained is the semantics of FEIF logic. This semantics can be formulated in different ways. One transparent one is to use infinite methods. This means replacing in effect a

quantified formula in the scope of \neg by a conjunction or disjunction of its substitution instances with respect to the names of all the members of the given domain. These will be infinite if the domain is infinite.

For instance, $\neg(\forall x)A[x]$ is true iff the disjunction

$$(5) \neg A[b_1] \vee \neg A[b_2] \vee \dots$$

is true, b_i being all the members of the domain. This suffices for the semantics in question.

This way of formulating a truth condition for second-order sentences helps to show in what sense the step from EIF logic to a logic based on unlimited use of tertium non datur means a step from elementary logic to a nonfinitistic logic.

However, we might want to have an explicit truth condition specified independently of any particular domain. Such a truth condition can be formulated by means of FEIF logic itself. In IF logic a natural truth condition asserts the existence of Skolem functions for S . This truth condition is obtained by replacing each existentially quantified formula $(\exists x)F[x]$ by $F[f(y_1, y_2, \dots)]$ where f is a new (Skolem) function bound to an initial $(\exists f)$ and $(Qy_1)(Qy_2), \dots$ are all the quantifiers on which $(\exists x)$ depends in S . We can do the same when S is a FEIF sentence, except for subformulas of the form $\neg F$. Here the analysis of the truth-condition of a contradictorily negated sentence presented earlier in this essay shows what can be done. There it was seen that $\neg F$ is equivalent to a Σ_2^1 formula. Hence a subformula of the form $\neg F$ can in a truth condition be replaced by a formula of the following form:

$$(6) (\forall g_1)(\forall g_2)\dots(\exists f_1)(\exists f_2)\dots\neg F^*$$

where g_1, g_2, \dots are the Skolem functions for F (i.e. strategy functions for the verifier) and f_1, f_2, \dots are the analogous strategy functions for the falsifier. The nesting of these functions is determined by the dependence relations between the different quantifiers in F . F^* is the result of replacing quantified variables by the appropriate function terms. The output is equivalent to an ordinary second-order sentence (assuming that atomic sentences obey the law of excluded middle). This output will have to satisfy certain conditions. The functions $g_1, g_2, \dots, f_1, f_2, \dots$ in (5) can have as their arguments only functions terms formed by means of these functions and the first function g_1 which is always a constant function. What this means

is that the quantifiers in $\neg F$ are independent of all the quantifiers further outside. In linguistic terms this very nearly says that contradictory negation is a barrier for anaphora – a phenomenon that is in evidence in natural languages. This procedure eventually moves all occurrences of \neg to positions where they have only atomic sentences within their scope and where they can be interpreted trivially. Thus the procedure can be thought of as specifying the meaning of \neg .

Thus for instance a (sub) formula of the form

$$(7) \neg(\exists x)(\forall y)(\exists z)F[x,y,z]$$

will be replaced in our reduction by a formula of the form

$$(8) (\forall f_1)(\forall f_2)(\exists g)\neg F(f_1,g(f_1),f_2(f_1,g(f_1)))$$

(In this case the function f_1 is reduced to a constant.)

By repeated applications of this procedure, moving from outside in, we can formulate a second-order truth condition for each FEIF sentence. In the reduct, all contradictory negations \neg occur in the negation prenexes of atomic formulas. Hence by the earlier reduction it is equivalent to a FEIF sentence. The functions $g_1, g_2, \dots, f_1, f_2, \dots$ deputize quantifiers in $\neg F$. These quantifiers in turn codify moves made by players in the game $G(\neg F)$ connected with $\neg F$. Hence, only variables occurring in the arguments of these functions can be variables bound to other quantifiers in $\neg F$.

In other words, by translating this second-order formula back to FEIF first-order logic we obtain for suitable FEIF languages a truth condition expressible in the same language.

Putting all this together, we can prove strictly the possibility of nominalism in the foundations of mathematics. This result is not only of philosophical interest. It has major implications for the foundations of mathematics in general and even for mathematics itself. First-order axiomatic set theory becomes redundant and so does in principle higher-order logic, except perhaps as a convenient shorthand. Our working logic will look more like general topology than conventional first-order logic or conventional set theory. I will leave the rest of the resulting revolution to my audience to carry out.

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